Asymptotic Quantum State Discrimination

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Mathematical Formalism.

Definition 1.1 (Quantum States)

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- $\mathscr{H}$ (*Hilbert Space*)

Quantum States

Definition 1.3 (Quantum States)

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- $\widehat{\mathbf{A}} \in \mathcal{B}(\mathscr{H})$

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Quantum mechanical systems are described via *positive semidefinite* trace class operators of trace 1 acting in a Hilbert space.

- $\mathscr{H}$ (*Hilbert Space*)
- $\widehat{\mathbf{A}} \in \mathcal{B}(\mathscr{H})$
- $Tr\{\widehat{\mathbf{A}}\} := \sum_i \langle \psi_i, \widehat{\mathbf{A}} \psi_i \rangle$ (ψ_i an ONB in $\mathscr{H}$) (*Trace*)

DEFINITION 1.5 (QUANTUM STATES)

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Quantum States

Definition 1.6 (Quantum States)

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- $\|\widehat{\mathbf{A}}\|_1 := Tr\left\{\sqrt{\widehat{\mathbf{A}}^\dagger \widehat{\mathbf{A}}}\right\}$
- $\mathscr{L}(\mathscr{H}) := \left\{\widehat{\mathbf{A}} \in \mathcal{B}(\mathscr{H}) \ : \ \|\widehat{\mathbf{A}}\|_1 < \infty\right\}$

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- $\mathscr{L}(\mathscr{H}) := \left\{\widehat{\mathbf{A}} \in \mathcal{B}(\mathscr{H}) \ : \ \|\widehat{\mathbf{A}}\|_1 < \infty\right\}$
- $\mathcal{S}(\mathscr{H}) := \left\{\hat{\boldsymbol{\rho}} \in \mathscr{L}(\mathscr{H}) \ : \ \langle \psi, \hat{\boldsymbol{\rho}}\psi \rangle \geq 0 \text{ for any } \psi \in \mathscr{H}, \text{ and } Tr\{\hat{\boldsymbol{\rho}}\} = 1\right\}$ (density operators)

Quantum State (contiued)

Definition 1.8 (Quantum States (continued))

Quantum State (contiued)

Definition 1.9 (Quantum States (continued))

- $\hat{\boldsymbol{\rho}}_\psi^2 = \hat{\boldsymbol{\rho}}_\psi$ (pure state)
  (Projector onto subspace of normalized state ψ)

Visual Computing

DEFINITION 1.10 (QUANTUM STATES (CONTINUED))

- $\hat{\boldsymbol{\rho}}_\psi^2 = \hat{\boldsymbol{\rho}}_\psi$ (pure state)
  (Projector onto subspace of normalized state ψ)
- $\sum_i p_i \hat{\boldsymbol{\rho}}_i$ with $\sum_{i=1} p_i = 1$(Quantum Mixture)

Dynamics and Quantum Maps

Definition 2.1 (Quatum map)

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two arbitrary Hilbert spaces. We define a quantum map $\mathscr{E}_t$ as a map from the set of density operators of the input space $\mathcal{S}(\mathscr{H}_1)$ to the set of density operators for the output space $\mathcal{S}(\mathscr{H}_2)$, with the following three axiomatic properties.

Visual Computing

DEFINITION 2.2 (QUATUM MAP)

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two arbitrary Hilbert spaces. We define a quantum map $\mathscr{E}_t$ as a map from the set of density operators of the input space $\mathcal{S}(\mathscr{H}_1)$ to the set of density operators for the output space $\mathcal{S}(\mathscr{H}_2)$, with the following three axiomatic properties.

- A1: $Tr\{\mathscr{E}_t(\hat{\boldsymbol{\rho}})\}$ is the probability that the process $\mathscr{E}$ occurs, when the $\hat{\boldsymbol{\rho}}$ is in the initial state. Thus, $0 \leq Tr\{\mathscr{E}_t(\hat{\boldsymbol{\rho}})\} \leq 1$ for any state $\hat{\boldsymbol{\rho}}$.

Definition 2.3 (Quatum map)

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two arbitrary Hilbert spaces. We define a quantum map $\mathscr{E}_t$ as a map from the set of density operators of the input space $\mathcal{S}(\mathscr{H}_1)$ to the set of density operators for the output space $\mathcal{S}(\mathscr{H}_2)$, with the following three axiomatic properties.

- A1: $Tr\{\mathscr{E}_t(\hat{\rho})\}$ is the probability that the process $\mathscr{E}$ occurs, when the $\hat{\rho}$ is in the initial state. Thus, $0 \leq Tr\{\mathscr{E}_t(\hat{\rho})\} \leq 1$ for any state $\hat{\rho}$.
- A2: $\mathscr{E}_t$ is a *convex-linear map* on the set of density operators, i.e. for a probability distribution $\{p_i\}$,

$$\mathscr{E}_t\left(\sum_i p_i \hat{\rho}_i\right) = \sum_i p_i \mathscr{E}_t(\hat{\rho}_i) \tag{1}$$

Definition 2.4 (Quantum map (continued))

- A3: $\mathscr{E}_t$ is a *completely positive* map. i.e., if $\mathscr{E}_t$ maps density operators of $\mathcal{S}(\mathscr{H}_1)$ to density operators of $\mathcal{S}(\mathscr{H}_2)$, then $\mathscr{E}_t(\widehat{\mathbf{A}})$ must be positive for any positive operator $\widehat{\mathbf{A}}$. Furthermore, let $\mathscr{H}_3$ a third arbitrary Hilbert space. It must then be true that $(\mathcal{I} \otimes \mathscr{E}_t)(\widehat{\mathbf{A}})$ is positive for any positive operator $\widehat{\mathbf{A}} \in \mathcal{B}(\mathscr{H}_1 \otimes \mathscr{H}_3)$ where $\mathcal{I}$ is the identity map on $\mathcal{B}(\mathscr{H}_1)$.
- When a quantum map preserves trace we call it a *quantum channel*.

Example 2.1 (Unitary Quantum Maps)

Let $\widehat{\mathbf{B}}$ be some self-adjoint operator acting in $\mathscr{H}$. Then the map $\mathscr{U}_t$ defined as follows a unitary quantum map. (A quantum Channel as a matter of fact!)

$$\mathscr{U}_t(\hat{\boldsymbol{\rho}}) := e^{-it\widehat{\mathbf{B}}}\hat{\boldsymbol{\rho}}e^{it\widehat{\mathbf{B}}} \tag{2}$$

Theorem 2.1 (Quantum Channel Representation, W. F. Stinespring (1955), M.-D. Choi (1975))

A trace-preserving map $\mathscr{E}_t$ is a *quantum channel* (i.e. trace-preserving) if and only if

$$\mathscr{E}_t(\hat{\boldsymbol{\rho}}) = \sum_i \widehat{\mathbf{M}}_i \hat{\boldsymbol{\rho}} \widehat{\mathbf{M}}_i^\dagger, \tag{3}$$

for some set of operators $\{\widehat{\mathbf{M}}_i\}_i$ which map the input Hilbert to the output Hilbert space, and $\sum_i \widehat{\mathbf{M}}_i^\dagger \widehat{\mathbf{M}}_i = \mathbb{I}$

Definition 2.5 (POVM)

Consider an arbitrary Hilbert space $\mathscr{H}$. A POVM is a set of semi-definite operators $\left\{\widehat{\mathbf{M}}_i^\dagger \widehat{\mathbf{M}}_i\right\}_i$ acting in $\mathscr{H}$ that sum to the identity operator. i.e.

$$\sum_i \widehat{\mathbf{M}}_i^\dagger \widehat{\mathbf{M}}_i = \mathbb{I}_{\mathscr{H}} \tag{4}$$

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The POVM may consist of an uncountable set of semi-definite operators as well. In such a case the analogous set of operators, e.g. $\widehat{\mathbf{M}}_x$ $(x \in \mathbb{R})$ must meet the same constraint. i.e.

$$\int \widehat{\mathbf{M}}_x^\dagger \widehat{\mathbf{M}}_x dx = \mathbb{I}_{\mathscr{H}} \tag{5}$$

Quantum Measurement

POVM and Quantum Measurement (continued)

Definition 3.1 (Quantum Measurement)

Let $\{\widehat{\mathbf{M}}_i^\dagger \widehat{\mathbf{M}}_i\}_i$ be a POVM $\widehat{\mathbf{B}}\left(\mathscr{H}\right)$ and $\hat{\boldsymbol{\rho}}$ some state. Given a quantum system in state $\hat{\boldsymbol{\rho}}$, the theory of quantum probability treats $\widehat{\mathbf{M}}_i^\dagger \widehat{\mathbf{M}}_i$ as events, while the traces $p_i := Tr\{\hat{\boldsymbol{\rho}} \widehat{\mathbf{M}}_i^\dagger \widehat{\mathbf{M}}_i\}$ are postulated to be the probabilities of the i the event occurring after conducting a measurement on the system. The operators $\widehat{\mathbf{M}}_i$ are known as measurement operators.

Definition 3.2 (Quantum Measurement)

Let $\{\widehat{\mathbf{M}}_i^\dagger \widehat{\mathbf{M}}_i\}_i$ be a POVM $\widehat{\mathbf{B}}\left(\mathscr{H}\right)$ and $\hat{\rho}$ some state. Given a quantum system in state $\hat{\rho}$, the theory of quantum probability treats $\widehat{\mathbf{M}}_i^\dagger \widehat{\mathbf{M}}_i$ as events, while the traces $p_i := Tr\{\hat{\rho}\widehat{\mathbf{M}}_i^\dagger \widehat{\mathbf{M}}_i\}$ are postulated to be the probabilities of the i the event occurring after conducting a measurement on the system. The operators $\widehat{\mathbf{M}}_i$ are known as measurement operators. If one conducts

a measurement on the quantum state $\hat{\rho}$ and the outcome is that which is indexed by i, then the post-measurement state is postulated to be

$$\frac{\widehat{\mathbf{M}}_i \hat{\rho} \widehat{\mathbf{M}}_i^\dagger}{Tr\left\{\widehat{\mathbf{M}}_i \hat{\rho} \widehat{\mathbf{M}}_i^\dagger\right\}} \tag{6}$$

Definition 3.3 (Quantum Measurement)

The state above is the resulting state assuming that one has "read out" the measurement. However, if one does not read out the results of the measurement, what one has is a mixture

$$\sum_i p_i \frac{\widehat{\mathbf{M}}_i \hat{\rho} \widehat{\mathbf{M}}_i^\dagger}{Tr\{\widehat{\mathbf{M}}_i \hat{\rho} \widehat{\mathbf{M}}_i^\dagger\}} \tag{7}$$

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Given that $p_i = Tr\{\widehat{\mathbf{E}}_i \hat{\rho}\}$, the unread state of the system is

$$\sum_i p_i \frac{\widehat{\mathbf{M}}_i \hat{\rho} \widehat{\mathbf{M}}_i^\dagger}{Tr\{\widehat{\mathbf{M}}_i \hat{\rho} \widehat{\mathbf{M}}_i^\dagger\}} = \sum_i Tr\{\widehat{\mathbf{M}}_i \hat{\rho} \widehat{\mathbf{M}}_i^\dagger\} \frac{\widehat{\mathbf{M}}_i \hat{\rho} \widehat{\mathbf{M}}_i^\dagger}{Tr\{\widehat{\mathbf{M}}_i \hat{\rho} \widehat{\mathbf{M}}_i^\dagger\}} = \tag{8}$$

$$\sum_i \widehat{\mathbf{M}}_i \hat{\rho} \widehat{\mathbf{M}}_i^\dagger (A \quad Quatum \quad Map!) \tag{9}$$

Quantum State Discrimination

Definition 4.1 (QSD)

Given a mixed state

$$\hat{\boldsymbol{\rho}} = \sum_{i=1}^{N} p_i \hat{\boldsymbol{\rho}}_i \tag{10}$$

where $\sum_{i=1}^{N} p_i = 1$, the theory of QSD aims to find a POVM $\{\widehat{\mathbf{E}}_l\}_{l=1}^{K} \subset \mathcal{B}(\mathscr{H})$ ($K \geq N$, $\widehat{\mathbf{E}}_l = \widehat{\mathbf{M}}_l^{\dagger}\widehat{\mathbf{M}}_l$) which minimizes the ojective function

$$p_E\left\{\{p_i, \hat{\boldsymbol{\rho}}_i\}_{i=1}^{N}, \{\widehat{\mathbf{M}}_l\}_{l=1}^{K}\right\} := 1 - \sum_{i=1}^{N} p_i \, Tr\{\widehat{\mathbf{M}}_i \hat{\boldsymbol{\rho}}_i \widehat{\mathbf{M}}_i^{\dagger}\} \tag{11}$$

(See "Quantum State Discrimination and its Applications" by Joonwoo Bae for an overview)

- Exact solution to $\min_{POVM} p_E\Big\{\{p_i, \hat{\boldsymbol{\rho}}_i\}_{i=1}^{N}, \{\widehat{\mathbf{M}}_l\}_{l=1}^{K}\Big\}$ is know only in a few special cases. E.g. "The Hellstrom bound"

QSD Continued

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- We are interested in the case where the $\hat{\boldsymbol{\rho}}_i(t)$ are dynamic and in the long term behaviour wrt t.

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- In general computational optimization techniques are utilized for estimating the QSD problem.
- We are interested in the case where the $\hat{\boldsymbol{\rho}}_i(t)$ are dynamic and in the long term behaviour wrt t.
- When is Asymptotic QSD $\lim_{t\to} \min_{POVM} p_E\{\{p_i, \hat{\boldsymbol{\rho}}_i\}_{i=1}^N, \{\widehat{\mathbf{M}}_l\}_{l=1}^K\} = 0??$

Some important results in QSD

Theorem 4.1 (D. Qiu and L. Li Bound (2008))

$$\min_{POVM} p_E \geq \frac{1}{2}\left(1 - \frac{1}{2(N-1)} \sum_i \sum_{j;j\neq i} \left\| p_i \hat{\rho}_i - p_j \hat{\rho}_j \right\|_1 \right) \tag{12}$$

Theorem 4.4 (D. Qiu and L. Li Bound (2008))

$$\min_{POVM} p_E \geq \frac{1}{2}\Big(1 - \frac{1}{2(N-1)} \sum_{i} \sum_{j;j\neq i} \big\| p_i \hat{\boldsymbol{\rho}}_i - p_j \hat{\boldsymbol{\rho}}_j \big\|_1 \Big) \tag{12}$$

Theorem 4.5 (A. Montanaro Bound (2008))

$$\min_{POVM} p_E \geq \frac{1}{2} \sum_{i} \sum_{j;j\neq i} p_i p_j \Big\| \sqrt{\hat{\boldsymbol{\rho}}_i} \sqrt{\hat{\boldsymbol{\rho}}_j} \Big\|_1^2 \tag{13}$$

Theorem 4.7 (D. Qiu and L. Li Bound (2008))

$$\min_{POVM} p_E \geq \frac{1}{2}\left(1 - \frac{1}{2(N-1)} \sum_{i} \sum_{j;j\neq i} \left\| p_i \hat{\rho}_i - p_j \hat{\rho}_j \right\|_1\right) \tag{12}$$

Theorem 4.8 (A. Montanaro Bound (2008))

$$\min_{POVM} p_E \geq \frac{1}{2} \sum_{i} \sum_{j;j\neq i} p_i p_j \left\| \sqrt{\hat{\rho}_i} \sqrt{\hat{\rho}_j} \right\|_1^2 \tag{13}$$

Theorem 4.9 (E. Knill and H. Barnum (2002))

$$\min_{POVM} p_E \leq \sum_{i} \sum_{j;j\neq i} \sqrt{p_i p_j} \left\| \sqrt{\hat{\rho}_i} \sqrt{\hat{\rho}_j} \right\|_1 \tag{14}$$

Asymptotic Quantum State Discrimination

Definition 5.1 (Asymptotic Quantum State Discrimination (AQSD))

Given a mixture $\sum_i \mathscr{E}_{i,t}(\hat{\boldsymbol{\rho}})$,AQSD is the problem of computing/estimating$\lim_{t\rightarrow} \min_{POVM} p_E\{\{p_i, \hat{\boldsymbol{\rho}}_i\}_{i=1}^N, \{\widehat{\mathbf{M}}_l\}_{l=1}^K\}$

Definition 5.3 (Asymptotic Quantum State Discrimination (AQSD))

Given a mixture $\sum_i \mathscr{E}_{i,t}(\hat{\boldsymbol{\rho}})$,AQSD is the problem of computing/estimatinglim$_{t\to}$ min$_{POVM}$ $p_E\{\{p_i, \hat{\boldsymbol{\rho}}_i\}_{i=1}^N, \{\widehat{\mathbf{M}}_l\}_{l=1}^K\}$

Definition 5.4 (The AQSD Full Solvability Problem)

Given a mixture $\sum_i \mathscr{E}_{i,t}(\hat{\boldsymbol{\rho}})$, what are necessary and sufficient conditions for $\lim_{t\to} \min_{POVM} p_E\{\{p_i, \hat{\boldsymbol{\rho}}_i\}_{i=1}^N, \{\widehat{\mathbf{M}}_l\}_{l=1}^K\} = 0$ to occur (*fully solvable*)?

Let us specialize to the following mixture.

Calculation 5.1 (Bounding)

$$\sum_{i=1}^{N} p_i e^{-itx_i \widehat{\mathbf{B}}} \hat{\boldsymbol{\rho}}_{\psi} e^{itx_i \widehat{\mathbf{B}}} \tag{15}$$

Let us specialize to the following mixture.

Calculation 5.2 (Bounding)

$$\sum_{i=1}^{N} p_i e^{-itx_i\widehat{\mathbf{B}}}\hat{\boldsymbol{\rho}}_{\psi}e^{itx_i\widehat{\mathbf{B}}} \tag{15}$$

using the Knill-Barnum and Montanaro bounds we have

Let us specialize to the following mixture.

Calculation 5.3 (Bounding)

$$\sum_{i=1}^{N} p_i e^{-itx_i\widehat{\mathbf{B}}}\hat{\boldsymbol{\rho}}_{\psi} e^{itx_i\widehat{\mathbf{B}}} \tag{15}$$

using the Knill-Barnum and Montanaro bounds we have

$$\sum_{i}\sum_{j;j\neq i} p_i p_j \left\| \sqrt{e^{-itx_i\widehat{\mathbf{B}}}\hat{\boldsymbol{\rho}}_{\psi} e^{itx_i\widehat{\mathbf{B}}}} \sqrt{e^{-itx_j\widehat{\mathbf{B}}}\hat{\boldsymbol{\rho}}_{\psi} e^{itx_j\widehat{\mathbf{B}}}} \right\|_1^2 \leq \tag{16}$$

$$p_E\left\{\{p_i, e^{-itx_i\widehat{\mathbf{B}}}\hat{\boldsymbol{\rho}}_{\psi} e^{itx_i\widehat{\mathbf{B}}}\}_{i=1}^{N}, \{\widehat{\mathbf{M}}_l\}_{l=1}^{K}\right\} \leq \tag{17}$$

$$\leq \sum_{i}\sum_{j;j\neq i} \sqrt{p_i p_j} \left\| \sqrt{e^{-itx_i\widehat{\mathbf{B}}}\hat{\boldsymbol{\rho}}_{\psi} e^{itx_i\widehat{\mathbf{B}}}} \sqrt{e^{-itx_j\widehat{\mathbf{B}}}\hat{\boldsymbol{\rho}}_{\psi} e^{itx_j\widehat{\mathbf{B}}}} \right\|_1 \tag{18}$$

Calculation 5.4 (Bounding)

$$\sum_i \sum_{j;j\neq i} p_i p_j \left|\langle\psi, e^{-it(x_i-x_j)\widehat{\mathbf{B}}}\psi\rangle\right|^2 \leq \tag{19}$$

$$p_E\left\{\{p_i, e^{-itx_i\widehat{\mathbf{B}}}\hat{\boldsymbol{\rho}}_\psi e^{itx_i\widehat{\mathbf{B}}}\}_{i=1}^N, \{\widehat{\mathbf{M}}_l\}_{l=1}^K\right\} \leq \tag{20}$$

$$\leq \sum_i \sum_{j;j\neq i} \sqrt{p_i p_j}\left|\langle\psi, e^{-it(x_i-x_j)\widehat{\mathbf{B}}}\psi\rangle\right| \tag{21}$$

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$$p_E\left\{\{p_i, e^{-itx_i\widehat{\mathbf{B}}}\hat{\boldsymbol{\rho}}_\psi e^{itx_i\widehat{\mathbf{B}}}\}_{i=1}^N, \{\widehat{\mathbf{M}}_l\}_{l=1}^K\right\} \leq \tag{20}$$

$$\leq \sum_{i}\sum_{j;j\neq i} \sqrt{p_i p_j}\left|\langle \psi, e^{-it(x_i-x_j)\widehat{\mathbf{B}}}\psi\rangle\right| \tag{21}$$

When do we have

$$\lim_{t\to\infty}\left|\langle \psi, e^{-it(x_i-x_j)\widehat{\mathbf{B}}}\psi\rangle\right| = 0?? \tag{22}$$

Definition 5.5 (Rajchman Measure)

A finite Borel probability measure μ on **R** is called a Rajchman measure if it satisfies

$$\lim_{t \to \infty} \hat{\mu}(t) = 0 \tag{23}$$

where $\hat{\mu}(t) := \int_{\mathbb{R}} e^{2i\pi tx} d\mu(x) \quad , \; t \in \mathbb{R}.$

Theorem 5.1 (Rajchman Subspace)

Let $\widehat{\mathbf{A}}$ be a self-adjoint operator acting on some arbitrary Hilbert space $\mathscr{H}$, then the set of vectors in $\mathscr{H}$ for which the spectral measure is a Rajchman measure, i.e.

$$\mathscr{H}_{rc} := \left\{\epsilon \mid \lim_{t\to\infty} \langle \psi, e^{-it\widehat{\mathbf{A}}}\psi\rangle = 0\right\}, \tag{24}$$

is a closed subspace which is invariant under $e^{-is\widehat{\mathbf{A}}}$. ($\mathscr{H}_{rc} \subset \mathscr{H}_c$)

Theorem 5.2 (Rajchman Subspace)

Let $\widehat{\mathbf{A}}$ be a self-adjoint operator acting on some arbitrary Hilbert space $\mathscr{H}$, then the set of vectors in $\mathscr{H}$ for which the spectral measure is a Rajchman measure, i.e.

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is a closed subspace which is invariant under $e^{-is\widehat{\mathbf{A}}}$. $(\mathscr{H}_{rc} \subset \mathscr{H}_c)$

Note, $\langle\psi, e^{-it\widehat{\mathbf{A}}}\psi\rangle = \langle\psi, \int e^{-it\lambda} d\widehat{\mathbf{E}}_\lambda\psi\rangle = \int e^{-it\lambda} d\langle\psi, \widehat{\mathbf{E}}_\lambda\psi\rangle = \int e^{-it\lambda} d\|\widehat{\mathbf{E}}_\lambda\psi\|^2$. $\|\widehat{\mathbf{E}}_\lambda\psi\|^2$ is a Rajchman measure.

Proposition 5.1

Consider the model described in this section by the states (19). $\psi \in \mathscr{H}_{rc}$ corresponding to $\widehat{\mathbf{B}}$ iff

$$\lim_{t\to\infty} \min_{POVM} p_E\left\{\left\{p_i,\ e^{-itx_i\widehat{\mathbf{B}}}\hat{\boldsymbol{\rho}}_\psi e^{itx_i\widehat{\mathbf{B}}}\right\}_{i=1}^{N}, \left\{\widehat{\mathbf{M}}_l\right\}_{l=1}^{K}\right\} = 0 \quad (25)$$

More General Results

Theorem 6.1

Let $\mathscr{H}$ be infinite-dimensional Hilbert space. Let $\widehat{\mathbf{B}}$ be a self-adjoint operator acting in $\mathscr{H}$ with a non-empty Rajchman subspace. Furthermore, let $\hat{\boldsymbol{\rho}}_i := \sum_{j=1}^{M_i} \eta_{ij} \boldsymbol{\rho}_{\phi_{ij}}$ be finite mixtures in $\mathcal{S}(\mathscr{H})$ for each i. Then,

$$\lim_{t\to\infty} \min_{POVM} p_E\left\{\{p_i,\ e^{-itx_i\widehat{\mathbf{B}}}\hat{\boldsymbol{\rho}}_i e^{itx_i\widehat{\mathbf{B}}}\}_{i=1}^{N}, \{\widehat{\mathbf{M}}_l\}_{l=1}^{K}\right\} = 0 \quad (26)$$

iff all of the $\phi_{ij} \in \mathscr{H}_{rc}$ of $\widehat{\mathbf{B}}$.

Conjecture 6.1 (Conjecture)

Let $\mathscr{H}$ be infinite-dimensional Hilbert space. Let $\widehat{\mathbf{B}}$ be a self-adjoint operator acting in $\mathscr{H}$ with a non-empty Rajchman subspace. Furthermore, let $\hat{\boldsymbol{\rho}}_i$ be an arbitrary mixture in $\mathcal{S}(\mathscr{H})$ for all i. Then,

$$\lim_{t\to\infty} \min_{POVM} p_E \left\{ \{ p_i,\ e^{-itx_i\widehat{\mathbf{B}}} \hat{\boldsymbol{\rho}}_i e^{itx_i\widehat{\mathbf{B}}} \}_{i=1}^{N}, \{ \widehat{\mathbf{M}}_l \}_{l=1}^{K} \right\} = 0 \quad (27)$$

iff all of the $\hat{\boldsymbol{\rho}}_i \in \mathcal{S}(\mathscr{H}_{rc})$ of $\widehat{\mathbf{B}}$.

So far we have the $\rightarrow$ direction.