Spectrum Broadcast Structures for Continuous Variables

Alberto Acevedo

June 12th 2023

1 Work by Jarek et all

In recent times significant attention has been given to a family of multipartite states named *spectrum broadcast structures* (SBS) [27] [28] [29] [52]. Since its genesis, the theory of SBS has been used as a tool in the discipline of *Quantum Foundations*; particularly in the theories of Decoherence and Quantum Darwinism [39][47][51][31]. Recently Quantum Darwinism and SBS theory have been shown to be equivalent under certain technical assumptions[32]. Motivating the theory of Quantum Darwinism and the theory of SBS is the question of objectivity in the quantum world. To avoid philosophical contention [27] [28][29], provides a definition of objectivity motivated by properties of classical dynamical systems. The multipartite quantum mechanical state satisfying such properties is called a Spectrum Broadcast Structure. The definition of objectivity proposed in [27] is:

Definition 1. *A state of the system S exists objectively if many observers can find out the state of S independently, and without perturbing it.*

The way to introduce objectivity in quantum systems, proposed by Korbicz and his collaborators is the SBS, as defined below.

Definition 2. *SBS: A Spectrum Broadcast Structure is a joint state of a central system S and an environment E, consisting of sub-environments $E^1, E^2, ..., E^N$:*

$$\hat{\boldsymbol{\rho}} = \sum_i p_i |i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_i^{E^k} \tag{1}$$

where $\{|i\rangle\}_i$ is some basis in the system's space, p_i are probabilities, and all states $\hat{\boldsymbol{\rho}}_i^{E^k}$ are perfectly distinguishable in the following sense:

$$F^2\big(\hat{\boldsymbol{\rho}}_i^{E^k}, \hat{\boldsymbol{\rho}}_j^{E^k}\big) = 0 \tag{2}$$

for all $i \neq j$ and for all $k = 1, ..., N$. Where $F(..., ...)$ is the quantum fidelity defined as $F\big(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\sigma}}\big) := \big\|\sqrt{\hat{\boldsymbol{\rho}}}\sqrt{\hat{\boldsymbol{\sigma}}}\big\|_1$

In [27] it is proven that SBS satisfies the desired definition of objectivity and that it is the only such structure that satisfies such a definition. The challenge then becomes showing that typical multipartite states converge to SBS in the large time limit [29]. The principal models studied in SBS literature [27][28][29] are of the *quantum-measurement* limit type, meaning SBS that arise from dynamics generated by $Hamiltonians$ in which the interaction term between the system S and the environment E greatly dominates, i.e. $\hat{\mathbf{H}}_{tot} \approx \hat{\mathbf{H}}_I$ (tot means total and I indicates "interaction terms").

We consider a quantum system interacting with N macroscopic environments. We assume that the joint initial state has the product form:

$$\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\rho}}_{S_0} \otimes \bigotimes_{k=1}^{N} \hat{\boldsymbol{\rho}}^{E_0^k} \tag{3}$$

In the sate (3) we write the subscript 0 in E_0^k in order to emphasize that this is the initial state of the kth environment E^k, similarly, we use the subscript S_0 to highlight the initial state of the system.

We assume the *quantum-measurement limit*, $\hat{\mathbf{H}}_{tot} \approx \hat{\mathbf{H}}_I$. Hence

$$\hat{\mathbf{H}}_I = \gamma f(\hat{\mathbf{X}}) \otimes \sum_{k=1}^{N} g_k(\hat{\mathbf{B}}_k) \tag{4}$$

where the operators $\hat{\mathbf{X}}$ and $\hat{\mathbf{B}}_k$ above are the position operator and some arbitrary observable respectively; each acting on its respective space, i.e. all of the $\hat{\mathbf{B}}_k$ act on different Hilbert spaces.

The functions $f(x)$ and $g_k(x)$ are only assumed to be continuous. A Hamiltonian of the form (4) is said to be of the von Neumann type [48]. The corresponding time evolution operator is therefore

$$\hat{\mathbf{U}}_t = e^{-it\gamma f(\hat{\mathbf{X}})\otimes \sum_{k=1}^{N} g_k(\hat{\mathbf{B}}_k)}. \tag{5}$$

We evolve our total initial state using the evolution operator (5).

$$\hat{\boldsymbol{\rho}}_t = \left(e^{-it\gamma f(\hat{\mathbf{X}})\otimes \sum_{k=1}^{N} g_k(\hat{\mathbf{B}}_k)}\right)\hat{\boldsymbol{\rho}}_{S_0} \otimes \bigotimes_{k=1}^{N} \hat{\boldsymbol{\rho}}^{E_0^k}\left(e^{it\gamma f(\hat{\mathbf{X}})\otimes \sum_{k=1}^{N} g_k(\hat{\mathbf{B}}_k)}\right). \tag{6}$$

To study the state of the subsystem formed by the system S and the first N_E environments, we take the partial trace of the time-evolved density operator over the remaining $M_E := N - N_E$ environments.

Claim 1. *Partial trace:*

$$Tr_{E^{N_E+1},E^{N_E+2},\ldots,E^N}\{\hat{\boldsymbol{\rho}}_t\} = \mathscr{U}_{N_E,t}\left(\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_0}) \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}^{E_0^k}\right) \tag{7}$$

using the notation

$$Tr_{A,B}\{\ldots\} := Tr_A\{Tr_B\{\ldots\}\}. \tag{8}$$

Where

$$\mathscr{U}_{n,t}(\hat{\mathbf{A}}) := e^{-it\gamma f(\hat{\mathbf{X}})\otimes\hat{\mathbf{S}}_n}(\hat{\mathbf{A}})e^{it\gamma f(\hat{\mathbf{X}})\otimes\hat{\mathbf{S}}_n} \tag{9}$$

$$\hat{\mathbf{S}}_n := \sum_{k=1}^{n} g_k(\hat{\mathbf{B}}_k)$$

and

$$\mathscr{E}_t^{M_E}\{\hat{\boldsymbol{\sigma}}\} := \int\int \langle x|\hat{\boldsymbol{\sigma}}|y\rangle \Gamma_{M_E}(t,x,y)|x\rangle\langle y|dxdy. \tag{10}$$

where

$$\Gamma_{M_E}(t,x,y) := \prod_{k=N_E+1}^{N} Tr_k\left\{\left(e^{-it\gamma f(x)g_k(\hat{\mathbf{B}}_k)}\right)\hat{\boldsymbol{\rho}}^{E_0^k}\left(e^{it\gamma f(y)g_k(\hat{\mathbf{B}}_k)}\right)\right\} \tag{11}$$

$M_E = N - N_E$, the number of traces being taken in equation (11).

Proof.

$$Tr_{E_{N_E+1},E_{N_E+2},\ldots,E_N}\{\hat{\boldsymbol{\rho}}_t\} = \tag{12}$$

$$Tr_{E_{N_E+1},E_{N_E+2},\ldots,E_N}\left\{\left(e^{-it\gamma f(\hat{X})\otimes \sum_{k=1}^{N} g_k(\hat{\mathbf{B}}_k)}\right)\hat{\boldsymbol{\rho}}_{S_0} \otimes \bigotimes_{k=1}^{N} \hat{\boldsymbol{\rho}}^{E_0^k}\left(e^{it\gamma f(\hat{X})\otimes \sum_{k=1}^{N} g_k(\hat{\mathbf{B}}_k)}\right)\right\} = \tag{13}$$

$$\mathscr{U}_{N_E,t}\left(Tr_{E_{N_E+1},E_{N_E+2},\ldots,E_N}\left\{\left(e^{-it\gamma f(\hat{\mathbf{X}})\otimes \sum_{k=N_E+1}^{N} g_k(\hat{\mathbf{B}}_k)}\right)\hat{\boldsymbol{\rho}}_{S_0} \otimes \bigotimes_{k=N_E+1}^{N} \hat{\boldsymbol{\rho}}^{E_0^k}\left(e^{it\gamma f(\hat{\mathbf{X}})\otimes \sum_{k=N_E+1}^{N} g_k(\hat{\mathbf{B}}_k)}\right)\right\}\bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}^{E_0^k}\right) \tag{14}$$

Let us now use the generalized eigenvectors of $\hat{\mathbf{X}}$ in order to write $\hat{\boldsymbol{\rho}}_{S_0} = \int\int K_S(x,y)|x\rangle\langle y|dxdy$ where $K_S(x,y) = \langle x|\hat{\boldsymbol{\rho}}|y\rangle$. Using the latter,

$$e^{-it\gamma f(\hat{\mathbf{X}})\otimes \sum_{k=N_E+1}^{N} g_k(\hat{\mathbf{B}}_k)}\hat{\boldsymbol{\rho}}_{S_0} \otimes \bigotimes_{k=N_E+1}^{N} \hat{\boldsymbol{\rho}}^{E_0^k} e^{it\gamma f(\hat{\mathbf{X}})\otimes \sum_{k=N_E+1}^{N} g_k(\hat{\mathbf{B}}_k)} = \tag{15}$$

$$\int\int K_S(x,y)|x\rangle\langle y|\left(e^{-it\gamma f(x)\sum_{k=N_E+1}^{N} g_k(\hat{\mathbf{B}}_k)}\left(\bigotimes_{k=N_E+1}^{N} \hat{\boldsymbol{\rho}}^{E_0^k}\right)e^{it\gamma f(y)\sum_{k=N_E+1}^{N} g_k(\hat{\mathbf{B}}_k)}\right)dxdy = \tag{16}$$

$$\int\int K_S(x,y)|x\rangle\langle y| \otimes \bigotimes_{k=N_E+1}^{N} e^{-it\gamma f(x) g_k(\hat{\mathbf{B}}_k)} \hat{\boldsymbol{\rho}}^{E_0^k} e^{it\gamma f(y) g_k(\hat{\mathbf{B}}_k)} dxdy \tag{17}$$

Furthermore

$$Tr_{E_{N_E+1},E_{N_E+2},\ldots,E_N}\left\{\int\int K_S(x,y)|x\rangle\langle y| \otimes \bigotimes_{k=N_E+1}^{N} e^{-it\gamma f(x) g_k(\hat{\mathbf{B}}_k)} \hat{\boldsymbol{\rho}}^{E_0^k} e^{it\gamma f(y) g_k(\hat{\mathbf{B}}_k)} dxdy\right\} = \tag{18}$$

$$\int\int K_S(x,y)|x\rangle\langle y| Tr_{E_{N_E+1},E_{N_E+2},\ldots,E_N}\left\{\bigotimes_{k=N_E+1}^{N} e^{-it\gamma f(x) g_k(\hat{\mathbf{B}}_k)} \hat{\boldsymbol{\rho}}^{E_0^k} e^{it\gamma f(y) g_k(\hat{\mathbf{B}}_k)}\right\} dxdy = \tag{19}$$

$$\int\int K_S(x,y)\Gamma_{M_E}(t,x,y)|x\rangle\langle y| dxdy = \mathscr{E}_t^{M_E}\left(\hat{\boldsymbol{\rho}}_{S_0}\right) \tag{20}$$

Finally, using (14) and (20) we have

$$(14) = \mathscr{U}_{N_E,t}\left(\mathscr{E}_t^{M_E}\left(\hat{\boldsymbol{\rho}}_{S_0}\right) \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}^{E_0^k}\right) \tag{21}$$

□

To simplify the notation, we shall forgo all but two macroscopic environments, i.e.$N = 2$, $N_f = m = 1$. After partial tracing over one of the environments we obtain the following density operator.

$$\hat{\boldsymbol{\rho}}_t := \mathscr{U}_{1,t}\big(\mathscr{E}_t^1\{\hat{\boldsymbol{\rho}}_{S_0}\} \otimes \hat{\boldsymbol{\rho}}^{E_0^1}\big). \tag{22}$$

The map $\mathscr{E}_t^1$ is a decoherence quantum map and $\mathscr{U}_{1,t}$ is a unitary map obtained from the Hamiltonian (4) for the case $N = 2$. In [29] an SBS state associated with (22) is defined for every value of $t > 0$; it is shown there that (22) converges to this state (in the trace norm, see below), as t goes to ∞. Its form may be deduced from (22) as follows. We rewrite (22) as:

$$\hat{\boldsymbol{\rho}}_t = \mathscr{U}_{1,t}\big(\mathscr{E}_t^1\{\hat{\boldsymbol{\rho}}_{S_0}\} \otimes \hat{\boldsymbol{\rho}}^{E_0^1}\big) = \sum_{i,j=1}^{d_S} \sigma_{i,j}\gamma_{i,j}^2(t)|i\rangle\langle j| \otimes \hat{\boldsymbol{\rho}}_{x_i,x_j}^{E_t^1} \tag{23}$$

where $\{|i\rangle\}_{i=1}^{d_s}$ are the eigenvectors of $\hat{\mathbf{X}}$, with corresponding eigenvalues $\{x_i\}_{i=1}^{d_S}$ ($\hat{\mathbf{X}}$ in [29] has discrete spectrum). Where we have used the definitions

$$\hat{\boldsymbol{\rho}}_{x,y}^{E_t^k} := e^{-it\gamma f(x) g_k(\hat{\mathbf{B}}_k)} \hat{\boldsymbol{\rho}}^{E_0^k} e^{itf(y) g_k(\hat{\mathbf{B}}_k)} \; (k = 1, 2) \tag{24}$$

$$\sigma_{i,j} := \langle i|\hat{\boldsymbol{\rho}}_{S_0}|i\rangle \tag{25}$$

$$\gamma_{i,j}^k(t) := Tr\big\{\hat{\boldsymbol{\rho}}_{x_i,x_j}^{E_t^k}\big\} \tag{26}$$

$$\hat{\boldsymbol{\rho}}_{x}^{E_t^k} := e^{-it\gamma f(x) g_k(\hat{\mathbf{B}}_k)} \hat{\boldsymbol{\rho}}^{E_0^k} e^{itf(x) g_k(\hat{\mathbf{B}}_k)} \; (k = 1, 2). \tag{27}$$

The SBS approximating (22) is defined by restricting the sum on the RHS of (23) to the diagonal terms—the terms with $i = j$. We will label it as follows.

$$\hat{\boldsymbol{\rho}}_{diag,t} := \sum_{i=1}^{d_S} \sigma_i\gamma_i^2(t)|i\rangle\langle i| \otimes \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} \tag{28}$$

Notice that for $i = j$, $\gamma_{i,j}(t) = 1$, so

$$\hat{\boldsymbol{\rho}}_{diag,t} = \sum_{i=1}^{d_S} \sigma_i|i\rangle\langle i| \otimes \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1}. \tag{29}$$

The next step is to choose for every t a projection-valued-measure (PVM) acting on the space $\mathscr{H}_S \otimes \mathscr{H}_{E_1}$ (For the case considered in [29], $dim(\mathscr{H}_S) = d_S < \infty$ and $dim(\mathscr{H}_{E_1}) = d_{E_1} < \infty$). To define such a PVM, the authors use the eigenbasis of the operator $\hat{\mathbf{X}}$: the elements of the PVM are of the form $|i\rangle\langle i| \otimes \hat{\mathbf{P}}_j^{E_t^1}$ where the $\{|i\rangle\langle i|\}_{i=1}^{d_S}$ and $\{\hat{\mathbf{P}}_j^{E_t^1}\}_{j=1}^{d_S} \cup \{\mathbb{I} - \sum_{i=1}^{d_S} \hat{\mathbf{P}}_i^{E_t^1}\}$ resolve the identity operators in $\mathcal{B}(\mathscr{H}_S)$ and $\mathcal{B}(\mathscr{H}_{E_1})$ respectively, so that, in particular,$\{\hat{\mathbf{P}}_j^{E_t^1}\}_{j=1}^{d_S} \cup \{\mathbb{I} - \sum_{i=1}^{d_S} \hat{\mathbf{P}}_i^{E_t^1}\}$ is a PVM in the environment's Hilbert space. The latter PVM is then used to approximate the (22) by an SBS state:

$$\hat{\boldsymbol{\rho}}_{SBS,t} := \frac{1}{\mathscr{N}} \sum_{j=1}^{d_S} \left(|j\rangle\langle j| \otimes \boldsymbol{P}_j^{E_t^1} \right) \hat{\boldsymbol{\rho}}_{diag,t} \left(|j\rangle\langle j| \otimes \hat{\mathbf{P}}_j^{E_t^1} \right) = \tag{30}$$

$$\sum_{i=1}^{d_S} \tilde{\sigma}_i |i\rangle\langle i| \otimes \left(\hat{\mathbf{P}}_i^{E_t^1} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} \hat{\mathbf{P}}_i^{E_t^1} \right). \tag{31}$$

Here $\mathscr{N}$ is a normalizing constant and $\tilde{\sigma}_i := \frac{\sigma_i}{\mathscr{N}}$. The operator (31) is indeed an SBS state. If (22) converges to an object with the form (31) as $t \to \infty$, we say that (22) is asymptotically SBS. Convergence is meant here in the sense of trace distance. Namely, one would like to show that

$$\frac{1}{2} \min_{PVM} \left\| \hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{SBS,t} \right\|_1 \to 0 \text{ as } t \to \infty \tag{32}$$

where for each t the minimization is taken over all projective-valued-measures $\{\hat{\mathbf{P}}_i^{E_t^1}\}_{i=1}^{d_S} \cup \{\mathbb{I} - \sum_{i=1}^{d_S} \hat{\mathbf{P}}_i^{E_t^1}\}$. An attempt is made in [29] to prove (32) but the argument provided there is incomplete. In what follows we discuss the bounds presented in [29], as well as propose and prove an alternative bound for the trace distance in (32).

In the paper [29], a bound is conjectured for the trace distance in (32). In the case of two environments (one of which is traced over), the bound becomes:

$$\frac{1}{2} \min_{PVM} \left\| \hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{SBS,t} \right\|_1 \leq \Gamma(t) + \sum_i \sum_{j;j \neq i} \sqrt{\sigma_i \sigma_j} F\left(\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1}, \hat{\boldsymbol{\rho}}_{x_j}^{E_t^1} \right) \tag{33}$$

where $\Gamma(t) := \sum_{i \neq j} |\sigma_{i,j} \gamma_{i,j}^2(t)|$. In general, for the case where M environmental degrees of freedom have been traced out and N_E remain, the bound looks as follows.

$$\frac{1}{2} \min_{PVM} \left\| \hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{SBS,t} \right\|_1 \leq \Gamma(t) + \sum_i \sum_{j;j \neq i} \sqrt{\sigma_i \sigma_j} \sum_{k=1}^{N_E} F\left(\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}, \hat{\boldsymbol{\rho}}_{x_j}^{E_t^k} \right) \tag{34}$$

where now, $\Gamma(t) = \sum_{i \neq j} |\sigma_{i,j}| \prod_{k=N+1}^{M+N} |\gamma_{i,j}^k(t)|$, where again $\gamma_{i,j}^k(t) = Tr\left[\hat{\boldsymbol{\rho}}_{x_i,x_j}^{E_t^k} \right]$. If true, this result would allow to estimate the minimum on the LHS, using the asymptotic properties of $\Gamma(t)$ and the fidelity terms in (34). As it is currently not known to be true, we will not be using it.

1.1 A new bound for the trace distance of a multipartite state and an approximating SBS state

In what follows we use an unnormalized version of (30): $\hat{\boldsymbol{\rho}}_{PSBS,t} := \mathscr{N} \hat{\boldsymbol{\rho}}_{SBS,t}$. This state is just the state (30) without the normalization factor $\frac{1}{\mathscr{N}}$. In practice it is easier to bound $\left\| \hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{PSBS,t} \right\|_1$ and then utilize Lemma 1, stated below, to bound $\left\| \hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{SBS,t} \right\|_1$.

Lemma 1. *$\|\hat{\boldsymbol{\rho}} - \eta \hat{\sigma}\|_1 \leq L$ implies $\|\hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\sigma}}\| \leq 2L$ for constants $L \geq 0$ and $\eta \in [0, 1]$*

Proof. Using reverse triangle inequality we see that

$$L \geq \|\hat{\boldsymbol{\rho}} - \eta \hat{\boldsymbol{\sigma}}\|_1 \geq \left| \|\hat{\boldsymbol{\rho}}\|_1 - \|\eta \hat{\boldsymbol{\sigma}}\|_1 \right| = \|\hat{\boldsymbol{\rho}}\|_1 - \|\eta \hat{\boldsymbol{\sigma}}\|_1 = 1 - \eta \tag{35}$$

furthermore

$$\|\hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\sigma}}\|_1 = \|\hat{\boldsymbol{\rho}} - \eta\hat{\boldsymbol{\sigma}} + \eta\hat{\boldsymbol{\sigma}} - \hat{\sigma}\|_1 \leq \|\hat{\boldsymbol{\rho}} - \eta\hat{\boldsymbol{\sigma}}\|_1 + \|\eta\hat{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}}\|_1 \leq \tag{36}$$

$$L + (1-\eta)\|\hat{\sigma}\|_1 = L + (1-\eta) \leq L + L = 2L \tag{37}$$

□

We now prove some preliminary inequalities.

$$\left\|\hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{PSBS,t}\right\|_1 = \tag{38}$$

$$\left\| \sum_{i,j=1}^{d_S} \sigma_{i,j}\gamma_{i,j}^2(t)|i\rangle\langle j| \otimes \hat{\boldsymbol{\rho}}_{x_i,x_j}^{E_t^1} - \sum_{i=1}^{d_S} \sigma_i |i\rangle\langle i| \otimes \hat{\mathbf{P}}_i^{E_t^1} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} \hat{\mathbf{P}}_i^{E_t^1} \right\|_1 \leq \tag{39}$$

$$\left\| \sum_{i=1}^{d_S} \sigma_i |i\rangle\langle i| \otimes \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} - \sum_{i=1}^{d_S} \sigma_i |i\rangle\langle i| \otimes \hat{\mathbf{P}}_i^{E_t^1} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} \hat{\mathbf{P}}_i^{E_t^1} \right\|_1 + \left\| \sum_i \sum_{j;j\neq i}^{d_S} \sigma_{i,j}\gamma_{i,j}^2(t)|i\rangle\langle j| \otimes \hat{\boldsymbol{\rho}}_{x_i,x_j}^{E_t^1} \right\|_1 \leq \tag{40}$$

$$\sum_{i=1}^{d_S} \left\| \sigma_i |i\rangle\langle i| \otimes \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} - \sigma_i |i\rangle\langle i| \otimes \hat{\mathbf{P}}_i^{E_t^1} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} \hat{\mathbf{P}}_i^{E_t^1} \right\|_1 + \left\| \sum_i \sum_{j\neq i}^{d_S} \sigma_{i,j}\gamma_{i,j}(t)|i\rangle\langle j| \otimes \hat{\boldsymbol{\rho}}_{x_i,x_j}^{E_t^1} \right\|_1 = \tag{41}$$

$$\sum_{i=1}^{d_S} \sigma_i \left\| |i\rangle\langle i| \otimes \left(\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} - \hat{\mathbf{P}}_i^{E_t^1} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} \hat{\mathbf{P}}_i^{E_t^1} \right) \right\|_1 + \left\| \sum_i \sum_{j;j\neq i}^{d_S} \sigma_{i,j}\gamma_{i,j}^2(t)|i\rangle\langle j| \otimes \hat{\boldsymbol{\rho}}_{x_i,x_j}^{E_t^1} \right\|_1 = \tag{42}$$

$$\sum_{i=1}^{d_S} \sigma_i \left\| \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} - \hat{\mathbf{P}}_i^{E_t^1} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} \hat{\mathbf{P}}_i^{E_t^1} \right\|_1 + \sum_i \sum_{j\neq i}^{d_S} |\sigma_{i,j}\gamma_{i,j}^2(t)| \tag{43}$$

We may therefore conclude that

$$\frac{1}{2} \min_{PVM} \left\|\hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{SBS,t}\right\|_1 \leq \tag{44}$$

$$\min_{PVM} \left(\sum_{i=1}^{d_S} \sigma_i \left\| \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} - \hat{\mathbf{P}}_i^{E_t^1} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} \hat{\mathbf{P}}_i^{E_t^1} \right\|_1 \right) + \sum_i \sum_{j;j\neq i}^{d_S} |\sigma_{i,j}\gamma_{i,j}^2(t)| \tag{45}$$

The second term in the sum (45) is the decoherence term which is independent of the choice of the PVM minimized over. The decoherence term is simple to study provided that we are able to compute the trace defining $\gamma_{i,j}(t)$. The first term in (45) involves a minimization over all PVM for each value of t. Rather than attempting to solve the minimization problem exactly, below we estimate the first term in (45) for a particular (judiciously chosen) PVM, obtaining an upper bound on the true minimum.

1.1.1 Bounding the variational term

Switching notation for a bit. Consider a mixed state of the form $\sum_{i=1}^N p_i\hat{\rho}_i$, where $\sum_{i=1}^N p_i = 1$ and the $\hat{\rho}_i$ are pure states in a Hilbert space of dimension greater than N, i.e. one-dimensional projections $|\psi_i\rangle\langle\psi_i|$, where $\{|\psi_i\rangle\}_{i=1}^N$ are normalized vectors. Assuming that $|\psi_i\rangle$ are linearly independent, we may use the well-known Gram-Schmidt procedure to define an associated orthonormal set.

Definition 3. *Gram-Schmidt Procedure: Assume that the set $\{|\psi\rangle_i\}_{i=1}^N$, of vectors in some vector space V, is a linearly independent set. Then the following construction yields an orthonormal set.*

$$|\phi_1\rangle = |\psi_1\rangle \tag{46}$$

$$|\phi_2\rangle = \frac{1}{\alpha_2}\left\{ |\psi_2\rangle - \langle\phi_1|\psi_2\rangle|\phi_1\rangle \right\} \tag{47}$$

$$\vdots$$

$$|\phi_N\rangle = \frac{1}{\alpha_N}\left\{|\psi_N\rangle - \sum_{k=1}^{N-1}\langle\phi_k|\psi_N\rangle|\phi_k\rangle\right\} \tag{48}$$

Here $\alpha_i := \||\psi_i\rangle - \sum_{k=1}^{i-1}\langle\phi_k|\psi_i\rangle|\phi_k\rangle\| = \sqrt{1 - \sum_{k=1}^{i-1}|\langle\phi_k|\psi_i\rangle|^2}$ *for* $i > 1$ *and* $\alpha_1 = 1$ *are the respective normalization constants. We have* $Span\left\{\{|\psi_i\rangle\}_{i=1}^N\right\} = Span\left\{\{|\phi_i\rangle\}_{i=1}^N\right\}$.

The orthonormal set $\{|\phi_i\rangle\}_{i=1}^N$ may be used for the construction of a PVM, namely

$$\big\{|\phi_i\rangle\langle\phi_i|\big\}_{i=1}^N \cup \left\{\mathbb{I} - \sum_{i=1}^N |\phi_i\rangle\langle\phi_i|\right\}. \tag{49}$$

We will use it to estimate $\min_{PVM}\sum_{i=1}^N p_i\big\|\hat{\boldsymbol{\rho}}_i - \hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{P}}_i\big\|_1$:

$$\min_{PVM}\sum_{i=1}^N p_i\big\|\hat{\boldsymbol{\rho}}_i - \hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{P}}_i\big\|_1 \le \sum_{i=1}^N p_i\big\|\hat{\boldsymbol{\rho}}_i - |\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i|\phi_i\rangle\langle\phi_i|\big\|_1. \tag{50}$$

Lemma 2. *Let* $\hat{\rho}_i$ *and* $|\phi_i\rangle$ *be defined as above; also let* $i > 1$, *then*

$$\big\|\hat{\boldsymbol{\rho}}_i - |\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i|\phi_i\rangle\langle\phi_i|\big\|_1 \le 2\sum_{k=1}^{i-1}|\langle\phi_k|\psi_i\rangle|$$

Proof.

$$\big\|\hat{\boldsymbol{\rho}}_i - |\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i|\phi_i\rangle\langle\phi_i|\big\|_1 = \big\||\psi_i\rangle\langle\psi_i|\hat{\boldsymbol{\rho}}_i|\psi_i\rangle\langle\psi_i| - |\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i|\phi_i\rangle\langle\phi_i|\big\|_1 =$$

$$\Big\|\Big(|\psi_i\rangle\langle\psi_i| - |\phi_i\rangle\langle\phi_i|\Big)\hat{\boldsymbol{\rho}}_i|\psi_i\rangle\langle\psi_i| + |\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i\Big(|\psi_i\rangle\langle\psi_i| - |\phi_i\rangle\langle\phi_i|\Big)\Big\|_1 \le$$

$$\Big\|\Big(|\psi_i\rangle\langle\psi_i| - |\phi_i\rangle\langle\phi_i|\Big)\hat{\boldsymbol{\rho}}_i|\psi_i\rangle\langle\psi_i|\Big\|_1 + \Big\||\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i\Big(|\psi_i\rangle\langle\psi_i| - |\phi_i\rangle\langle\phi_i|\Big)\Big\|_1 \le$$

$$\Big\||\psi_i\rangle\langle\psi_i| - |\phi_i\rangle\langle\phi_i|\Big\|_1\Big\|\hat{\boldsymbol{\rho}}_i|\psi_i\rangle\langle\psi_i|\Big\|_1 + \Big\||\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i\Big\|_1\Big\||\psi_i\rangle\langle\psi_i| - |\phi_i\rangle\langle\phi_i|\Big\|_1 =$$

$$\Big\||\psi_i\rangle\langle\psi_i| - |\phi_i\rangle\langle\phi_i|\Big\|_1\left(\Big\|\hat{\boldsymbol{\rho}}_i|\psi_i\rangle\langle\psi_i|\Big\|_1 + \Big\||\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i\Big\|_1\right) \le$$

$$\Big\||\psi_i\rangle\langle\psi_i| - |\phi_i\rangle\langle\phi_i|\Big\|_1\left(\big\|\hat{\boldsymbol{\rho}}_i\big\|_1\Big\||\psi_i\rangle\langle\psi_i|\Big\|_1 + \Big\||\phi_i\rangle\langle\phi_i|\Big\|_1\big\|\hat{\boldsymbol{\rho}}_i\big\|_1\right) \le$$

$$2\Big\||\psi_i\rangle\langle\psi_i| - |\phi_i\rangle\langle\phi_i|\Big\|_1 = 2\sqrt{1 - |\langle\psi_i|\phi_i\rangle|^2} =$$

$$2\sqrt{1 - \left|\frac{1}{\alpha_i}\left(1 - \sum_{k=1}^{i-1}|\langle\phi_k|\psi_i\rangle|^2\right)\right|^2} = 2\sqrt{1 - \left|\frac{\left(1 - \sum_{k=1}^{i-1}|\langle\phi_k|\psi_i\rangle|^2\right)}{\sqrt{\left(1 - \sum_{k=1}^{i-1}|\langle\phi_k|\psi_i\rangle|^2\right)}}\right|^2}$$

$$= 2\sqrt{1 - 1 + \sum_{k=1}^{i-1}|\langle\phi_k|\psi_i\rangle|^2} = 2\sqrt{\sum_{k=1}^{i-1}|\langle\phi_k|\psi_i\rangle|^2} \le 2\sum_{k=1}^{i-1}|\langle\phi_k|\psi_i\rangle|$$

where we have used the fact that $\sum_{k=1}^{i-1}|\langle\phi_k|\psi_i\rangle|^2 \le 1$ due to *Bessel's* inequality in the last line. □

The term $\sum_{k=1}^{i-1}|\langle\phi_k|\psi_i\rangle|$ may be understood by analyzing it through the scope of its related determinant. We present this result as a lemma.

Lemma 3.

$$|\phi_j\rangle = \frac{1}{\sqrt{D_{j-1}D_j}} \begin{vmatrix} \langle\psi_1|\psi_1\rangle & \langle\psi_1|\psi_2\rangle & \dots & \langle\psi_1|\psi_j\rangle \\ \langle\psi_2|\psi_1\rangle & \langle\psi_2|\psi_2\rangle & \dots & \langle\psi_2|\psi_j\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi_{j-1}|\psi_1\rangle & \langle\psi_{j-1}|\psi_2\rangle & \dots & \langle\psi_{j-1}|\psi_j\rangle \\ |\psi_1\rangle & |\psi_2\rangle & \dots & |\psi_j\rangle \end{vmatrix}$$

where

$$D_j := \begin{vmatrix} \langle\psi_1|\psi_1\rangle & \langle\psi_1|\psi_2\rangle & \dots & \langle\psi_1|\psi_j\rangle \\ \langle\psi_2|\psi_1\rangle & \langle\psi_2|\psi_2\rangle & \dots & \langle\psi_2|\psi_j\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi_j|\psi_1\rangle & \langle\psi_j|\psi_2\rangle & \dots & \langle\psi_j|\psi_j\rangle \end{vmatrix}$$

defining $|\phi_1\rangle := |\psi_1\rangle$, $D_0 := 1$ *and* $D_1 = 1$ *in order to make sense of the case* $j = 1$ *and* $k = 0, 1$ *for* $|\phi_i\rangle$ *and* D_k *respectively.*

In determinant form, $\langle\psi_i|\phi_k\rangle$ may now be written as follows.

$$\langle\psi_i|\phi_k\rangle = \frac{1}{\sqrt{D_{k-1}D_k}} \begin{vmatrix} \langle\psi_1|\psi_1\rangle & \langle\psi_1|\psi_2\rangle & \dots & \langle\psi_1|\psi_k\rangle \\ \langle\psi_2|\psi_1\rangle & \langle\psi_2|\psi_2\rangle & \dots & \langle\psi_2|\psi_k\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi_{k-1}|\psi_1\rangle & \langle\psi_{k-1}|\psi_2\rangle & \dots & \langle\psi_{k-1}|\psi_k\rangle \\ \langle\psi_i|\psi_1\rangle & \langle\psi_i|\psi_2\rangle & \dots & \langle\psi_i|\psi_k\rangle \end{vmatrix} \tag{51}$$

The power behind viewing the states $|\phi_i\rangle$ in their determinant form is that now we need only compute inner products between elements of the set $\{|\psi_i\rangle\}_{i=1}^N$ in order to estimate the effectiveness of the PVM (49) in approximating a solution for $\min_{PVM} \sum_{i=1}^N p_i \|\hat{\boldsymbol{\rho}}_i - \hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{P}}_i\|_1$. Recall that the states $\{|\psi_i\rangle\}_{i=1}^N$ are normalized. Furthermore, assume that $\langle\psi_i|\psi_j\rangle = \varepsilon_{ij}$ for all $i \neq j \in \{1, ..., N\}$, where ε_{ij} are complex numbers satisfying $|\varepsilon_{ij}| \leq \delta$ for all $i \neq j \in \{1, .., N\}$, where δ is small. Since, under this assumption, all entries of the last column of the matrix (51) are small, this also implies that $\left\|\hat{\boldsymbol{\rho}}_i - |\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i|\phi_i\rangle\langle\phi_i|\right\|_1$ is small for all i, thanks to Lemma 2.

The above estimates imply the following theorem.

Theorem 1. *Consider a mixed state of the form* $\sum_{i=1}^N p_i\hat{\boldsymbol{\rho}}_i$, $\sum_{i=1}^N p_i = 1$, *where* $\hat{\boldsymbol{\rho}}_i := |\psi_i\rangle\langle\psi_i|$ *are pure states acting on a* Hilbert *space of dimension greater than* N. *Furthermore, assume that the states* $\{|\psi_i\rangle\}_i$ *are linearly independent. Then*

$$\min_{PVM} \sum_{i=1}^N p_i \|\hat{\boldsymbol{\rho}}_i - \hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{P}}_i\|_1 \leq$$

$$2\sum_{i=2}^N p_i \sum_{k=1}^{i-1} \frac{1}{|D_{k-1}D_k|} \left| \begin{vmatrix} \langle\psi_1|\psi_1\rangle & \langle\psi_1|\psi_2\rangle & \dots & \langle\psi_1|\psi_k\rangle \\ \langle\psi_2|\psi_1\rangle & \langle\psi_2|\psi_2\rangle & \dots & \langle\psi_2|\psi_k\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi_{k-1}|\psi_1\rangle & \langle\psi_{k-1}|\psi_2\rangle & \dots & \langle\psi_{k-1}|\psi_k\rangle \\ \langle\psi_i|\psi_1\rangle & \langle\psi_i|\psi_2\rangle & \dots & \langle\psi_i|\psi_k\rangle \end{vmatrix} \right|$$

where again

$$D_k := \begin{vmatrix} \langle\psi_1|\psi_1\rangle & \langle\psi_1|\psi_2\rangle & \dots & \langle\psi_1|\psi_k\rangle \\ \langle\psi_2|\psi_1\rangle & \langle\psi_2|\psi_2\rangle & \dots & \langle\psi_2|\psi_k\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi_j|\psi_1\rangle & \langle\psi_j|\psi_2\rangle & \dots & \langle\psi_k|\psi_k\rangle \end{vmatrix}$$

Proof. The proof follows directly from Lemma 3 and Lemma 2, and the fact that for $i = 1$ the corresponding projector is simply $|\psi_i\rangle\langle\psi_i|$ making the $i = 1$ term zero. □

In order to apply Theorem 1 to estimate the first term in (45), we assume that the initial state $\hat{\boldsymbol{\rho}}^{E_0^1}$ is pure; it is under this assumption that we expect the joint system-environment state to converge to an SBS as $\to \infty$. The purity of $\hat{\boldsymbol{\rho}}^{E_0^1}$ furthermore implies that the operators $\hat{\boldsymbol{\rho}}_i^{E_t^1}$ are pure for all i since the evolution (27) preserves purity of states. We may thus represent them as

$$\hat{\boldsymbol{\rho}}_i^{E_t^1} = |\psi_{i,t}\rangle\langle\psi_{i,t}| \tag{52}$$

We now use Theorem 1 to estimate (45):

$$\frac{1}{2}\min_{PVM}\sum_{i=1}^{d_S}\sigma_i\left\|\hat{\boldsymbol{\rho}}_i^{E_t^1} - \hat{\mathbf{P}}_i^{E_t^1}\hat{\boldsymbol{\rho}}_i^{E_t^1}\hat{\mathbf{P}}_i^{E_t^1}\right\|_1 + \frac{1}{2}\sum_i\sum_{j;j\neq i}^{d_S}|\sigma_{i,j}\gamma_{i,j}(t)| \leq \tag{53}$$

$$\sum_{i=2}^{d_S}\sigma_i\sum_{k=1}^{i-1}\frac{1}{|D_{k-1,t}D_{k,t}|}\left|\begin{vmatrix} \langle\psi_{1,t}|\psi_{1,t}\rangle & \langle\psi_{1,t}|\psi_{2,t}\rangle & \dots & \langle\psi_{1,t}|\psi_{k,t}\rangle \\ \langle\psi_{2,t}|\psi_{1,t}\rangle & \langle\psi_{2,t}|\psi_{2,t}\rangle & \dots & \langle\psi_{2,t}|\psi_{k,t}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi_{k-1,t}|\psi_{1,t}\rangle & \langle\psi_{k-1,t}|\psi_{2,t}\rangle & \dots & \langle\psi_{k-1,t}|\psi_{k,t}\rangle \\ \langle\psi_{i,t}|\psi_{1,t}\rangle & \langle\psi_{i,t}|\psi_{2,t}\rangle & \dots & \langle\psi_{i,t}|\psi_{k,t}\rangle \end{vmatrix}\right| + \frac{1}{2}\sum_i\sum_{j;j\neq i}^{d_S}|\sigma_{i,j}\gamma_{i,j}(t)| \tag{54}$$

where

$$D_{k,t} := \begin{vmatrix} \langle\psi_{1,t}|\psi_{1,t}\rangle & \langle\psi_{1,t}|\psi_{2,t}\rangle & \dots & \langle\psi_{1,t}|\psi_{k,t}\rangle \\ \langle\psi_{2,t}|\psi_{1,t}\rangle & \langle\psi_{2,t}|\psi_{2,t}\rangle & \dots & \langle\psi_{2,t}|\psi_{k,t}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi_{j,t}|\psi_{1,t}\rangle & \langle\psi_{j,t}|\psi_{2,t}\rangle & \dots & \langle\psi_{k,t}|\psi_{k,t}\rangle \end{vmatrix} \tag{55}$$

Given that computing determinants is a difficult task, one might wonder if there is a way to avoid doing so via further bounding the term (54) with another term that does not involve determinants. It turns out that such an approach is possible and if the entries $\langle\psi_{k,t}|\psi_{l,t}\rangle$ are small enough, the process is even easier to handle. We will develop such an approach in the section (**??**).

1.2 Mixed environmental states

We wil call the first term in the sum (45) the super error of discriminating the mixture $\sum_i \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1}$ with the PVM $\{\hat{\mathbf{P}}_i^{E_t^1}\}_i$. It is called the super error because (45) bounds the discrimination error $P_E\{p_i,\ \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1},\ \hat{\mathbf{P}}_i^{E_t^1}\}$ as follows.

$$P_E\left\{p_i,\ \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1},\ \hat{\mathbf{P}}_i^{E_t^1}\right\} = \sum_{i=1}^{d_S}\sigma_i Tr\left\{\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} - \hat{\mathbf{P}}_i^{E_t^1}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1}\hat{\mathbf{P}}_i^{E_t^1}\right\} \leq \sum_{i=1}^{d_S}\sigma_i\left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} - \hat{\mathbf{P}}_i^{E_t^1}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1}\hat{\mathbf{P}}_i^{E_t^1}\right\|_1 \tag{56}$$

The theory we have developed so far considers only the case where $\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1}$ are pure states for all i. In this subsection, we will further develop the previous section by providing the analog to Theorem 1 for the case where the environmental degrees of freedom are mixed states.

Using a simpler indexing scheme, consider a mixed state of the form $\sum_{i=1}^N p_i\hat{\boldsymbol{\rho}}_i$, where $\sum_{i=1}^N p_i = 1$ and the $\hat{\boldsymbol{\rho}}_i$ are mixed states which we will express as $\hat{\boldsymbol{\rho}}_i = \sum_{k=1}^M \eta_k\hat{\boldsymbol{\rho}}_{ik}$ where all of the $\hat{\boldsymbol{\rho}}_{ik}$ are pure states and $\sum_{k=1}^M \eta_k = 1$. Consider the super quantum state discrimination problem (now omitting the limits of the sums)

$$\min_{POVM}\sum_i p_i Tr\left\{\hat{\boldsymbol{\rho}}_i - \hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{M}}_i^\dagger\right\}. \tag{57}$$

The latter item is bounded above by the minimization problem that we have been concerned with in the previous section, i.e. minimizing over all PVM as opposed to minimizing over all POVM in (57). In turn it is also bounded above by the super PVM quantum state discrimination error as seen in the following relationship.

$$\min_{POVM}\sum_i p_i Tr\left\{\hat{\boldsymbol{\rho}}_i - \hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{M}}_i^\dagger\right\} \leq \min_{PVM}\sum_i p_i Tr\left\{\hat{\boldsymbol{\rho}}_i - \hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{M}}_i^\dagger\right\} \leq \tag{58}$$

$$\min_{PVM} \sum_i p_i \left\| \hat{\boldsymbol{\rho}}_i - \hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}}_i \hat{\mathbf{M}}_i^\dagger \right\|_1 \tag{59}$$

This follows from the fact that all PVMs are POVMs, making the space over which the objective function is minimized smaller and therefore yielding a smaller minimum.

Using the following bound from [66] we will bound (58) and (59) from below.

Theorem 2. *Montanaro: Let $\sum_i p_i \hat{\boldsymbol{\rho}}_i$ be a mixture of quantum states $\hat{\rho}_i$ where $\sum_i p_i = 1$. Then, for any POVM $\{\hat{\mathbf{M}}_i^\dagger \hat{\mathbf{M}}_i\}_i$,*

$$P_E\Big\{p_i,\ \hat{\boldsymbol{\rho}}_i,\ \hat{\mathbf{M}}_i^\dagger \hat{\mathbf{M}}_i\Big\} \geq \sum_{i>j} p_i p_j F(\hat{\boldsymbol{\rho}}_i, \hat{\boldsymbol{\rho}}_j) \tag{60}$$

We now have,

$$\sum_{i>j} p_i p_j F(\hat{\boldsymbol{\rho}}_i, \hat{\boldsymbol{\rho}}_j) \leq \min_{PVM} \sum_i p_i \left\| \hat{\boldsymbol{\rho}}_i - \hat{\mathbf{P}}_i \hat{\boldsymbol{\rho}}_i \hat{\mathbf{P}}_i \right\|_1 \tag{61}$$

Expanding the $\hat{\boldsymbol{\rho}}_i$ we see that

$$F(\hat{\boldsymbol{\rho}}_i, \hat{\boldsymbol{\rho}}_j) = F\Big(\sum_k \eta_k \hat{\boldsymbol{\rho}}_{ik}, \sum_k \eta_k \hat{\boldsymbol{\rho}}_{jk}\Big) \geq \sum_k \eta_k F\Big(\hat{\boldsymbol{\rho}}_{ik}, \hat{\boldsymbol{\rho}}_{jk}\Big) \tag{62}$$

where we have used the joint concavity of the fidelity [67] in the last line of (62).The bound (61) now implies that

$$\sum_{i>j} \sum_k p_i p_j \eta_k F\Big(\hat{\boldsymbol{\rho}}_{ik}, \hat{\boldsymbol{\rho}}_{jk}\Big) \leq \min_{PVM} \sum_i p_i \left\| \hat{\boldsymbol{\rho}}_i - \hat{\mathbf{P}}_i \hat{\boldsymbol{\rho}}_i \hat{\mathbf{P}}_i \right\|_1 \tag{63}$$

This inequality shows that a necessary condition for successful quantum state discrimination is that $\hat{\boldsymbol{\rho}}_{ik} \perp \hat{\boldsymbol{\rho}}_{jk}$ for all i, j, k where $i \neq j$. For the case where the $\hat{\boldsymbol{\rho}}_i$ are not mixed states the respective relationship implies that $\hat{\boldsymbol{\rho}}_i \perp \hat{\boldsymbol{\rho}}_j$ for $i \neq j$ which is what we expect from our analysis in the previous section. For the case of mixtures $\hat{\boldsymbol{\rho}}_i$ it is perhaps not surprising that we simply need to analyze the fidelities between elements of two different mixtures, say $\hat{\boldsymbol{\rho}}_i$ and $\hat{\boldsymbol{\rho}}_j$, in order to determine the discriminability of the mixture $\sum_{i=1}^N \hat{\boldsymbol{\rho}}_i$. As informative as (63) is, we have yet to learn anything about the constraints on fidelities involving multiple elements of the same mixture $\hat{\boldsymbol{\rho}}_i$, take $\hat{\boldsymbol{\rho}}_{ik}$ and $\hat{\boldsymbol{\rho}}_{il}$ for example. It could be the case that, in principle, there need not be any restrictions on said fidelities in order to attain successful quantum state discrimination but at the moment this is unknown to the authors.

We would like to bound (63) from above, and to do so we will once again take a constructive approach. The approach we will take shall be an adaptation of the methods employed in the proof of Theorem 1 and Lemma 2. Adapting the latter coupled with the fact that $\hat{\mathbf{P}}_i$ must be projectors will yield a bound that will be useful only for the cases where $\hat{\boldsymbol{\rho}}_{ik} \perp \hat{\boldsymbol{\rho}}_{jK}$ for all k when $i \neq j$ and $\hat{\boldsymbol{\rho}}_{ik} \perp \hat{\boldsymbol{\rho}}_{il}$ for all i when $l \neq k$.

Let us now construct a PVM that attempts to solve optimization on the right-hand side of inequality (61). We begin by noting that

$$\min_{PVM} \sum_i p_i \left\| \hat{\boldsymbol{\rho}}_i - \hat{\mathbf{P}}_i \hat{\boldsymbol{\rho}}_i \hat{\mathbf{P}}_i \right\|_1 \leq \min_{PVM} \sum_i \sum_k p_i \eta_k \left\| \hat{\boldsymbol{\rho}}_{ik} - \hat{\mathbf{P}}_i \hat{\boldsymbol{\rho}}_{ik} \hat{\mathbf{P}}_i \right\|_1 \tag{64}$$

This looks very similar to the PVM quantum state discrimination problem for pure states (note that $p_i \eta_k$ is a probability distribution) with the exception that now each element of the PVM $\big\{\hat{\mathbf{P}}_i\big\}_i$ corresponds to all elements $\hat{\boldsymbol{\rho}}_{ik}$. Following the methods from the previous section, one might suggest implementing the gram-schmidt procedure once more in order to obtain an orthonormal set of vectors $|\phi\rangle_i$, one for each i. However, in this case, the operators $\hat{\boldsymbol{\rho}}_i$ are mixed and therefore do not have a representation as a vector in an appropriate Hilbert space; being able to view the mixture $\sum_i \hat{\boldsymbol{\rho}}_i$ as an ensemble of pure states was one of the key assumptions that lead to Theorem 1. Perhaps there is a way to implement the Gram-Schmidt process to the end of producing an analog for Theorem (1) in a greater generality for the case where all of the $\hat{\boldsymbol{\rho}}_i$ are mixtures using the Hilbert-Schmidt inner product to generate orthogonal Hilbert-Schmidt operators in the Hilbert-Schmidt norm sense; however, the authors are unaware of any such approaches that have been met with success as of yet.

We now impose an assumption on the $\hat{\mathbf{P}}_i$ from (64) and once again highlights that the optimal $\hat{\mathbf{P}}_i$ need not have the following assumed structure.

$$\hat{\mathbf{P}}_i = \sum_{k=1}^{M} \hat{\mathbf{P}}_{ik} \tag{65}$$

One way to guarantee that a sum such as $\sum_{k=1}^{M} \hat{\mathbf{P}}_{ik}$ is a projector is to assume that $\hat{\mathbf{P}}_{ik}$ are all projectors with non-overlapping support.

Proof.

$$\hat{\mathbf{P}}_i^2 = \Big(\sum_{k=1}^{M} \hat{\mathbf{P}}_{ik}\Big)^2 = \sum_{k=1}^{M}\sum_{p=1}^{M} \hat{\mathbf{P}}_{ik}\hat{\mathbf{P}}_{ip} = \sum_{k=1}^{M}\sum_{p=1}^{M} \hat{\mathbf{P}}_{ik}\hat{\mathbf{P}}_{ip}\delta_{kp} = \sum_{k=1}^{M} \hat{\mathbf{P}}_{ik} = \hat{\mathbf{P}}_i \tag{66}$$

□

Since all of the $\hat{\boldsymbol{\rho}}_{ik}$ are pure states, we may apply the Gram-schmidt process as we have in the previous section in order to construct a PVM $\{\hat{\mathbf{P}}_{ik}\}_{ik}$. The resulting PVM elements $\hat{\mathbf{P}}_{ik}$ with the inclusion of the completion element $\mathbb{I} - \sum_i \sum_k \hat{\mathbf{P}}_{ik}$ form a PVM that resolves the identity. There are $N \times M$ states $\hat{\boldsymbol{\rho}}_{ik}$ since the index i ranges from 1 to N and the index k from 1 to M. Let us visualize the set of operators $\hat{\boldsymbol{\rho}}_{ik}$ as a matrix

$$\begin{pmatrix} \hat{\boldsymbol{\rho}}_{11} & \hat{\boldsymbol{\rho}}_{12} & \cdots & \hat{\boldsymbol{\rho}}_{1M} \\ \hat{\boldsymbol{\rho}}_{21} & \hat{\boldsymbol{\rho}}_{22} & \cdots & \hat{\boldsymbol{\rho}}_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\boldsymbol{\rho}}_{N1} & \hat{\boldsymbol{\rho}}_{N2} & \cdots & \hat{\boldsymbol{\rho}}_{NM} \end{pmatrix} \tag{67}$$

which we will flatten into the row vector

$$\vec{\mathscr{V}} := \begin{pmatrix} \hat{\boldsymbol{\rho}}_{11} & \ldots & \hat{\boldsymbol{\rho}}_{1M} & \hat{\boldsymbol{\rho}}_{21} & \ldots & \hat{\boldsymbol{\rho}}_{2M} & \ldots & \hat{\boldsymbol{\rho}}_{N1} & \ldots\ldots & \hat{\boldsymbol{\rho}}_{NM} \end{pmatrix}. \tag{68}$$

Let us now do a relabeling and call the sth component of $\mathscr{V}_s := |\xi_s\rangle\langle\xi_s|$. Given a specific value $s \in \{1, 2, ..., N \times M\}$ e can use the following formula to obtain the corresponding $\hat{\boldsymbol{\rho}}_{ik}$.

$$|\xi_s\rangle\langle\xi_s| = \hat{\boldsymbol{\rho}}_{\lceil \frac{s}{M} \rceil,\, s \bmod M}. \tag{69}$$

Assuming that the set $|\xi_s\rangle\langle\xi_s|$ is a linearly independent set we now apply the Gram-Schmidt process to obtain the family of orthonormal states

$$|\phi_1\rangle := |\xi_1\rangle \tag{70}$$

$$|\phi_s\rangle = \frac{1}{\alpha_s}\Big\{|\xi_s\rangle - \sum_{k=1}^{s-1}\langle\phi_k|\xi_s\rangle|\phi_k\rangle\Big\},\;\; s \in \{1, 2, ..., N \times M\} \tag{71}$$

where as before $\alpha_i := \big\||\xi_i\rangle - \sum_{k=1}^{i-1}\langle\phi_k|\xi_i\rangle|\phi_k\rangle\big\| = \sqrt{1 - \sum_{k=1}^{i-1}|\langle\phi_k|\xi_i\rangle|^2}$ for $i > 1$ and $\alpha_1 = 1$ are the respective normalization constants. An identity resolving PVM $\Big\{|\xi_s\rangle\langle\xi_s|\Big\}_s \bigcup \Big\{\mathbb{I} - \sum_s |\xi_s\rangle\langle\xi_s|\Big\}$ has been constructed, defining $\omega_s := p_{\lceil \frac{s}{M} \rceil}\eta_{s \bmod M}$ we may now rewrite $\sum_i \sum_k p_i\eta_k \Big\|\hat{\rho}_{ik} - P_i\hat{\rho}_{ik}P_i\Big\|_1$ as

$$\sum_s \omega_s \Big\||\xi_s\rangle\langle\xi_s| - \Big(\sum_{l=\lceil \frac{s}{M} \rceil}^{\lceil \frac{s}{M} \rceil + M} |\phi_l\rangle\langle\phi_l|\Big)|\xi_s\rangle\langle\xi_s|\Big(\sum_{l=\lceil \frac{s}{M} \rceil}^{\lceil \frac{s}{M} \rceil + M} |\phi_l\rangle\langle\phi_l|\Big)\Big\|_1 = \tag{72}$$

$$\sum_s \omega_s \Big\||\xi_s\rangle\langle\xi_s| - |\phi_s\rangle\langle\phi_s|\xi_s\rangle\langle\xi_s|\phi_s\rangle\langle\phi_s| - \Big(\sum_{l=\lceil \frac{s}{M} \rceil; l\neq s}^{\lceil \frac{s}{M} \rceil + M} |\phi_l\rangle\langle\phi_l|\Big)|\xi_s\rangle\langle\xi_s|\Big(\sum_{l=\lceil \frac{s}{M} \rceil; l\neq s}^{\lceil \frac{s}{M} \rceil + M} |\phi_l\rangle\langle\phi_l|\Big)\Big\|_1 \leq \tag{73}$$

$$\sum_s \omega_s \Big\| |\xi_s\rangle\langle\xi_s| - |\phi_s\rangle\langle\phi_s|\xi_s\rangle\langle\xi_s|\phi_s\rangle\langle\phi_s| \Big\|_1 + \sum_s \omega_s \Big\| \Big(\sum_{l=\lceil \frac{s}{M} \rceil; l\neq s}^{\lceil \frac{s}{M} \rceil + M} |\phi_l\rangle\langle\phi_l| \Big) |\xi_s\rangle\langle\xi_s| \Big(\sum_{l=\lceil \frac{s}{M} \rceil; l\neq s}^{\lceil \frac{s}{M} \rceil + M} |\phi_l\rangle\langle\phi_l| \Big) \Big\|_1 \leq \tag{74}$$

$$\sum_s \omega_s 2 \sum_{k=1}^{s-1} |\langle\phi_k|\xi_s\rangle| + \sum_s \omega_s \sum_{l=\lceil \frac{s}{M} \rceil; l\neq s}^{\lceil \frac{s}{M} \rceil + M} \sum_{k=\lceil \frac{s}{M} \rceil; k\neq s}^{\lceil \frac{s}{M} \rceil + M} \Big\| |\phi_l\rangle\langle\phi_l|\xi_s\rangle\langle\xi_s|\phi_k\rangle\langle\phi_k| \Big\|_1 = \tag{75}$$

$$\sum_s \omega_s 2 \sum_{k=1}^{s-1} |\langle\phi_k|\xi_s\rangle| + \sum_s \omega_s \left(\sum_{l=\lceil \frac{s}{M} \rceil; l\neq s}^{\lceil \frac{s}{M} \rceil + M} |\langle\phi_l|\xi_s\rangle| \right)^2 \tag{76}$$

where we have used Lemma 2 in going from (74) to (75). Using Lemma (3) we may explicitly write the terms $|\langle\phi_l|\xi_s\rangle$ as Gram-Schmidt determinants and use these to estimate the efficacy of the PVM built from (70).

In this paper, mixed environmental states as the environmental degrees of freedom are not the central focus. We shall therefore forego further analyzing the bound (76) at the moment and leave this for future work. However, we will point out that (76) may be further bounded by the following term.

$$(76) \leq 3 \sum_s \omega_s \sum_{l; l\neq s} |\langle\phi_l|\xi_s\rangle| \tag{77}$$

where the only restriction on the sums is that $l \neq s$. This may be better estimated using Lemma 3.

1.3 Further bounds for Theorem 1.

To begin we introduce three results that we shall be using.

Theorem 3. *Hadamard's inequality for determinants [13]: Let $\hat{\mathbf{A}}$ be some arbitrary $N \times N$ matrix with entries $A_{i,j}$. Then*

$$\det\big(\hat{\mathbf{A}}\big) \leq \prod_{j=1}^{N} \left(\sum_{i=1}^{N} |A_{ij}|^2 \right)^{\frac{1}{2}}.$$

Theorem 4. *[13] Let $\mathbb{I} + \hat{\mathbf{B}}$ be an $N \times N$ matrix with entries $\delta_{ij} + B_{ij}$ where $B_{i,i} = 0$ for all i. Then*

$$\det(\mathbb{I} + \hat{\mathbf{B}}) = \prod_{j=1}^{N} \big(1 + \lambda_j(\hat{B})\big)$$

Theorem 5. *Gerschgorin Theorem [53]: Let $\hat{\mathbf{A}}$ be an arbitrary $N \times N$ matrix with matrix elements $A_{i,j}$. Now, define*

$$\mathscr{D}_i := \Big\{ z \in \mathbb{C} : |z - A_{ii}| \leq \sum_{j; j\neq i} |A_{ij}| \Big\}.$$

Then, all of the eigenvalues of the operator $\hat{A}$ are found in the set $\mathscr{G}_N := \bigcup_{i=1}^{N} \mathscr{D}_i$. The sets $\mathscr{D}_i$ are known as Gerschgorin discs.

Now we use these theorems to prove the following.

Theorem 6.

$$\frac{1}{2} \min_{PVM} \sum_{i=1}^{d_S} \sigma_i \Big\| \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} - \mathbf{P}_i^{E_t^1} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} \mathbf{P}_i^{E_t^1} \Big\|_1 \leq \tag{78}$$

$$d_S \Big(1 + d_S M_{d_S} \Big)^{d_S - 1} \sum_{i\neq j}^{d_S} \frac{\sigma_i |\langle\psi_{i,t}|\psi_{j,t}\rangle|}{\min_{x\in\mathscr{G}_{d_S}} \big|1 - |x|\big|^{i-1}} \tag{79}$$

where

$$\mathscr{G}_k := \bigcup_{i=1}^{k} \mathscr{D}_i^k \tag{80}$$

$$\mathscr{D}_i^k := \left\{ x \in \mathbb{R} : |x| \leq \sum_{j;j\neq i} |B_{ij,t}^k| \right\} \; i \in \{1, ..., k\} \tag{81}$$

$$M_{d_S}(t) := \max_{n\neq m;\{1,...,d_S\}} |\langle \psi_{i,t}|\psi_{j,t}\rangle| \tag{82}$$

and

$$\hat{\mathbf{B}}_t^k := \begin{pmatrix} 0 & \langle \psi_{1,t}|\psi_{2,t}\rangle & \dots & \langle \psi_{1,t}|\psi_{k,t}\rangle \\ \langle \psi_{2,t}|\psi_{1,t}\rangle & 0 & \dots & \langle \psi_{2,t}|\psi_{k,t}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \psi_{j,t}|\psi_{1,t}\rangle & \langle \psi_{j,t}|\psi_{2,t}\rangle & \dots & 0 \end{pmatrix} \tag{83}$$

Proof. Assume that $k > 2$. Then, using Theorem 3

$$\hat{\mathbf{A}} := \left| \begin{vmatrix} \langle \psi_{1,t}|\psi_{1,t}\rangle & \langle \psi_{1,t}|\psi_{2,t}\rangle & \dots & \langle \psi_{1,t}|\psi_{k,t}\rangle \\ \langle \psi_{2,t}|\psi_{1,t}\rangle & \langle \psi_{2,t}|\psi_{2,t}\rangle & \dots & \langle \psi_{2,t}|\psi_{k,t}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \psi_{k-1,t}|\psi_{1,t}\rangle & \langle \psi_{k-1,t}|\psi_{2,t}\rangle & \dots & \langle \psi_{k-1,t}|\psi_{k,t}\rangle \\ \langle \psi_{i,t}|\psi_{1,t}\rangle & \langle \psi_{i,t}|\psi_{2,t}\rangle & \dots & \langle \psi_{i,t}|\psi_{k,t}\rangle \end{vmatrix} \right| \leq \tag{84}$$

$$\prod_{n=1}^{k} \left(\sum_{m=1}^{k} |A_{n,m}|^2 \right)^{\frac{1}{2}} \tag{85}$$

Where

$$A_{nm} = \langle \psi_{n,t}|\psi_{m,t}\rangle \;\; for \;\; n \in \{1, ..., k-1\} \;\; m \in \{1, ..., k\}$$

and

$$A_{nm} = \langle \psi_{i,t}|\psi_{m,t}\rangle \;\; for \;\; n = k \;\; m \in \{1, ..., k\}.$$

Therefore,

$$\prod_{n=1}^{k} \left(\sum_{m=1}^{k} |A_{nm}|^2 \right)^{\frac{1}{2}} = \prod_{n=1}^{k-1} \left(\sum_{m=1}^{k} |\langle \psi_{n,t}|\psi_{m,t}\rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{m=1}^{k} |\langle \psi_{i,t}|\psi_{m,t}\rangle|^2 \right)^{\frac{1}{2}} \leq \tag{86}$$

$$\left(\max_{n\in\{1,...,k-1\}} \sum_{m=1}^{k} |\langle \psi_{n,t}|\psi_{m,t}\rangle|^2 \right)^{\frac{k-1}{2}} \left(\sum_{m=1}^{k} |\langle \psi_{i,t}|\psi_{m,t}\rangle|^2 \right)^{\frac{1}{2}} \tag{87}$$

$$\left(1 + \max_{n\in\{1,...,k-1\}} \sum_{m=1;m\neq n}^{k} |\langle \psi_n(t)|\psi_m(t)\rangle|^2 \right)^{\frac{k-1}{2}} \left(\sum_{m=1}^{k} |\langle \psi_{i,t}|\psi_{m,t}\rangle|^2 \right)^{\frac{1}{2}} \leq \tag{88}$$

$$\left(1 + \max_{n\in\{1,...,d_S-1\}} \sum_{m=1;m\neq n}^{d_S} |\langle \psi_{n,t}|\psi_{m,t}\rangle|^2 \right)^{\frac{d_S-1}{2}} \left(\sum_{m=1}^{k} |\langle \psi_{i,t}|\psi_{m,t}\rangle|^2 \right)^{\frac{1}{2}} \leq \tag{89}$$

$$\left(1 + d_S \max_{n\neq m;\{1,...,d_S\}} |\langle \psi_{n,t}|\psi_{m,t}\rangle| \right)^{d_S-1} \left(\sum_{m=1}^{k} |\langle \psi_{i,t}|\psi_{m,t}\rangle| \right) \tag{90}$$

Now, let us shift our attention to the terms $D_{k,t}$ in Theorem 1, here time time-dependent.

$$D_{k,t} := \begin{vmatrix} \langle \psi_{1,t}|\psi_{1,t}\rangle & \langle \psi_{1,t}|\psi_{2,t}\rangle & \dots & \langle \psi_{1,t}|\psi_{k,t}\rangle \\ \langle \psi_{2,t}|\psi_{1,t}\rangle & \langle \psi_{2,t}|\psi_{2,t}\rangle & \dots & \langle \psi_{2,t}|\psi_{k,t}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \psi_j(t)|\psi_1(t)\rangle & \langle \psi_{j,t}|\psi_{2,t}\rangle & \dots & \langle \psi_{k,t}|\psi_{k,t}\rangle \end{vmatrix} \tag{91}$$

Using Theorem 4 we have that

$$|D_{k,t}| = \left| \prod_{j=1}^{k} \left(1 + \lambda_j\left(\hat{\mathbf{B}}_t^k\right)\right) \right| \tag{92}$$

where again

$$\hat{\mathbf{B}}_t^k := \begin{pmatrix} 0 & \langle\psi_{1,t}|\psi_{2,t}\rangle & \dots & \langle\psi_{1,t}|\psi_{k,t}\rangle \\ \langle\psi_{2,t}|\psi_{1,t}\rangle & 0 & \dots & \langle\psi_{2,t}|\psi_{k,t}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi_{j,t}|\psi_{1,t}\rangle & \langle\psi_{j,t}|\psi_{2,t}\rangle & \dots & 0 \end{pmatrix} \tag{93}$$

Now, using Theorem 5 we know that the eigenvalues of $\hat{B}_t^k$ lie within the Gerschgorin discs

$$\mathscr{D}_i^k := \left\{ x \in \mathbb{R} : |x| \leq \sum_{j;j\neq i} |B_{ij,t}| \right\} \ i \in \{1, ..., k\} \tag{94}$$

where we have replaced the $\mathbb{C}$ for $\mathbb{R}$ since Hermitian operators have real eigenvalues; we have also made use of the fact that $B_{ii,0}^k = 0$ for all i. The superscript of $\mathscr{D}_i^k$ is used to highlight its pertinence to the determinant $D_{k,t}$. Now,

$$|D_{k,t}| = \left| \prod_{j=1}^{k} \left(1 + \lambda_j\left(\hat{\mathbf{B}}_t^k\right)\right) \right| = \prod_{j=1}^{k} \left|1 + \lambda_j\left(\hat{\mathbf{B}}_t^k\right)\right| \geq \tag{95}$$

$$\geq \prod_{j=1}^{k} \min_{x\in\mathscr{G}_k} \left|1 + x\right| = \min_{x\in\mathscr{G}_k} \left|1 + x\right|^k. \tag{96}$$

Here we remind the reader that $\mathscr{G}_k := \bigcup_{i=1}^k \mathscr{D}_i^k$. Minimizing over a larger set yields a smaller minimum, hence,

$$\min_{x\in\mathscr{G}_k} \left|1 + x\right|^k \geq \min_{x\in\mathscr{G}_{d_S}} \left|1 + x\right|^k \geq \min_{x\in\mathscr{G}_{d_S}} \left|1 - |x|\right|^k. \tag{97}$$

Using (90), and (97) we may now further bound the determinant-including terms in result (54) to obtain

$$\frac{1}{2} \min_{PVM} \sum_{i=1}^{d_S} \sigma_i \left\| \hat{\boldsymbol{\rho}}_i^{E_t^1} - \hat{\mathbf{P}}_i^{E_t^1} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} \hat{\mathbf{P}}_i^{E_t^1} \right\|_1 \leq \tag{98}$$

$$\sum_{i=2}^{d_S} \sigma_i \sum_{k=1}^{i-1} \frac{\left(1 + d_S M_{d_s}(t)\right)^{d_S-1} \left(\sum_{m=1}^{k} |\langle\psi_{i,t}|\psi_{m,t}\rangle|\right)}{\min_{x\in\mathscr{G}_{d_S}} \left|1 - |x|\right|^k} = \tag{99}$$

$$\left(1 + d_S M_{d_S}(t)\right)^{d_S-1} \sum_{i=2}^{d_S} \sigma_i \sum_{k=1}^{i-1} \sum_{m=1}^{k} \frac{|\langle\psi_{i,t}|\psi_{m,t}\rangle|}{\min_{x\in\mathscr{G}_{d_S}} \left|1 - |x|\right|^k} \leq \tag{100}$$

$$\left(1 + d_S M_{d_S}(t)\right)^{d_S-1} \sum_{i=2}^{d_S} \sigma_i \sum_{k=1}^{i-1} \sum_{m=1}^{i-1} \frac{|\langle\psi_{i,t}|\psi_{m,t}\rangle|}{\min_{x\in\mathscr{G}_{d_S}} \left|1 - |x|\right|^{i-1}} \leq \tag{101}$$

$$\left(1 + d_S M_{d_S}(t)\right)^{d_S-1} \sum_{i=2}^{d_S} \sigma_i (i-1) \sum_{m=1}^{i-1} \frac{|\langle\psi_{i,t}|\psi_{m,t}\rangle|}{\min_{x\in\mathscr{G}_{d_S}} \left|1 - |x|\right|^{i-1}} \leq \tag{102}$$

$$d_S \left(1 + d_S M_{d_S}(t)\right)^{d_S-1} \sum_{i=2}^{d_S} \sigma_i \sum_{m=1}^{i-1} \frac{|\langle\psi_{i,t}|\psi_{m,t}\rangle|}{\min_{x\in\mathscr{G}_{d_S}} \left|1 - |x|\right|^{i-1}} \leq \tag{103}$$

$$d_S \left(1 + d_S M_{d_S}(t)\right)^{d_S-1} \sum_{i} \sum_{j;j\neq i}^{d_S} \frac{\sigma_i |\langle\psi_{i,t}|\psi_{j,t}\rangle|}{\min_{x\in\mathscr{G}_{d_S}} \left|1 - |x|\right|^{i-1}} \tag{104}$$

□

Corollary 1. *Assume that $d_S M_{d_S}(t) < 1$, then*

$$\frac{1}{2}\min_{PVM}\sum_{i=1}^{d_S}\sigma_i\left\|\hat{\boldsymbol{\rho}}_i^{E_t^1} - \hat{\mathbf{P}}_i^{E_t^1}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1}\hat{\mathbf{P}}_i^{E_t^1}\right\|_1 \leq \tag{105}$$

$$\frac{d_S\Big(1 + d_S M_{d_S}(t)\Big)^{d_S-1}}{\Big(1 - d_S M_{d_S}(t)\Big)^{d_S-1}}\sum_{i\neq j}^{d_S}\sigma_i|\langle\psi_{i,t}|\psi_{j,t}\rangle| \tag{106}$$

Corollary (1) may be generalized with ease to cases with N_E greater than 1.

2 Problems in SBS when introducing continuous variables

There are problems that arise when attempting to define an SBS state for the case where continuous variables are involved. To appreciate them, let us examine the state (22) in such a case. The system's state is now a density operator $\hat{\boldsymbol{\rho}}_{S_0}$ in an infinite-dimensional Hilbert space; for our purposes, it will be convenient to represent this space as $L^2(\mathbb{R})$. Analogously to (4), we define the interaction of the system with the environment as

$$H_I = \gamma\hat{\mathbf{X}}\otimes\hat{\mathbf{B}}$$

for simplicity; where $\hat{\mathbf{X}}$ is the position operator. Being a trace-class operator, $\hat{\boldsymbol{\rho}}_{S_0}$ can be represented as an integral operator, whose kernel we denote by $K(x,y)$. The expansion analogous to (23) is the following:

$$\hat{\boldsymbol{\rho}}_t = \int\int dxdyK(x,y)\gamma_{x,y}^2(t)|x\rangle\langle y|\otimes\hat{\boldsymbol{\rho}}_{x,y}^{E_t^1} \tag{107}$$

where as expected $\hat{\boldsymbol{\rho}}_{x,y}^{E_t^1} := e^{-ix\hat{\mathbf{B}}t}\hat{\boldsymbol{\rho}}^{E_0^1}e^{iy\hat{\mathbf{B}}t}$ and $\gamma_{x,y}^2(t) := Tr\{\hat{\boldsymbol{\rho}}_{x,y}^{E_t^2}\}$. Unlike the state (22), the state (107) does not have a clear decomposition into off-diagonal and of diagonal terms using the spectral decomposition of the operator $\hat{\boldsymbol{X}}$ in terms of generalized eigenvectors $|x\rangle$, which we have employed to expand $\mathscr{U}_{1,t}\big(\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_0})\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big) = \big(e^{-it\hat{\mathbf{X}}\otimes\hat{\mathbf{B}}}\big)\big(\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{s_0})\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)\big(e^{-it\hat{\mathbf{X}}\otimes\hat{\mathbf{B}}}\big)$. In the finite-dimensional case, we could clearly distinguish between diagonal and off-diagonal entries in order to deduce an SBS structure approximating the state in question. In the continuous variable case, this approach breaks down since the "diagonal" term is now

$$\hat{\boldsymbol{\rho}}_t = \int dxK(x,x)|x\rangle\langle x|\otimes\hat{\boldsymbol{\rho}}_x^{E_t^1} \tag{108}$$

This is not a trace class operator, since it is unitarily equivalent to a tensor product of a multiplication operator and a trace class operator—thus it cannot represent a quantum state.

Another difficulty in moving into the continuous variable case is an increase in complexity when dealing with trace norms; starting from the fact that $\big\||x\rangle\langle y|\big\|_1$ is undefined for generalized states $|x\rangle$ and $|y\rangle$.

3 SBS for continuous variables

We now discuss the phenomenon of decoherence which results from an evolution of the system under a quantum map. Let us focus on the case described by (22), where the system's state evolves according to:

$$\hat{\boldsymbol{\rho}}_t := \big(e^{-it\gamma f(\hat{\mathbf{X}})\otimes g(\hat{\mathbf{B}})}\big)\big(\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_0})\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)\big(e^{it\gamma f(\hat{\mathbf{X}})\otimes g(\hat{\mathbf{B}})}\big) := \mathbf{U}_{1,t}\big(\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_0})\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big) \tag{109}$$

We will be assuming the states $\hat{\boldsymbol{\rho}}_{S_0}$ and $\hat{\boldsymbol{\rho}}^{E_0^1}$ are pure. Under our assumptions, the operators $\hat{\mathbf{X}}$ and $\hat{\mathbf{B}}$ are self-adjoint and have purely absolutely continuous spectrum. As we have done in earlier sections, we write $\hat{\boldsymbol{\rho}}_{S_0} = \int\int dxdyK(x,y)|x\rangle\langle y|$ using the resolution of identity associated with the operator $\hat{\mathbf{X}}$, i.e. $\hat{\mathbf{X}} = \int x|x\rangle\langle x|dx$. Furthermore, the quantum map $\mathscr{E}_t$ has the representation

$$\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}_{s_0}\big) = \int\int K(x,y)\Gamma(t,x,y)\big|x\big\rangle\big\langle y\big|dxdy \tag{110}$$

where $\Gamma(t,x,y)$ is a kernel yielding non-unitary dynamics obtained via partial tracing as seen in (11). Substituting this into (109) we obtain

$$\hat{\boldsymbol{\rho}}_t = \int\int dxdyK(x,y)\Gamma(t,x,y)\big|x\big\rangle\big\langle y\big| \otimes \hat{\boldsymbol{\rho}}^{E_t^1}_{x,y} \tag{111}$$

where we remind the reader that $\hat{\boldsymbol{\rho}}^{E_t^1}_{x,y} := e^{-it\gamma f(x)g(\hat{\mathbf{B}})}\hat{\boldsymbol{\rho}}^{E_0^1}e^{it\gamma f(y)g(\hat{\mathbf{B}})}$.

In what follows we will partition the real line into intervals of length greater than or equal to some resolution limit Σ, which may in general depend on t but we will suppress this dependence for now and discuss such cases in Section 4. The idea is using a net $\{x_i\}_i$ of the real line and building the sets $\Delta_i := (x_i - \frac{\sigma_i}{2}, x_i + \frac{\sigma_i}{2})$ of lenght σ_i constrained to the criteria $x_{i+1} - x_i \geq \Sigma\ \forall i$, $\sigma_i \geq \Sigma$ for all integers i. We index the σ_i because in general, this partition need not be made of sets of equal length.

It will be of interest to us to estimate the trace norm of the non-diagonal terms of the operator (111), i.e. $i \neq j$. We will use this partitioning scheme to rewrite (111) in an equivalent form as follows.

$$\hat{\boldsymbol{\rho}}_t = \sum_{i,j}\int_{\Delta_i}\int_{\Delta_j} dxdyK(x,y)\Gamma(t(x-y))|x\rangle\langle y| \otimes \hat{\boldsymbol{\rho}}^{E_t^1}_{x,y} \tag{112}$$

Some elementary work leads to

$$\Bigg\|\sum_i\sum_{j;j\neq i}\int_{\Delta_i}\int_{\Delta_j} dxdyK(x,y)\Gamma(t,x,y)|x\rangle\langle y| \otimes \hat{\boldsymbol{\rho}}^{E_t^1}_{x,y}\Bigg\|_1 = \tag{113}$$

$$\Bigg\|e^{-it\gamma f(\hat{X})\otimes g(\hat{\mathbf{B}})}\Bigg(\Bigg\{\sum_i\sum_{j;j\neq i}\int_{\Delta_i}\int_{\Delta_j} dxdyK(x,y)\Gamma(t,x,y)|x\rangle\langle y|\Bigg\} \otimes \hat{\boldsymbol{\rho}}^{E_0^1}\Bigg)e^{it\gamma f(\hat{X})\otimes g(\hat{\mathbf{B}})}\Bigg\|_1 = \tag{114}$$

$$\Bigg\|\Bigg\{\sum_i\sum_{j;j\neq i}\int_{\Delta_i}\int_{\Delta_j} dxdyK(x,y)\Gamma(t,x,y)|x\rangle\langle y|\Bigg\} \otimes \hat{\boldsymbol{\rho}}^{E_0^1}\Bigg\|_1 = \tag{115}$$

$$\Bigg\|\Bigg\{\sum_i\sum_{j;j\neq i}\int_{\Delta_i}\int_{\Delta_j} dxdyK(x,y)\Gamma(t,x,y)|x\rangle\langle y|\Bigg\}\Bigg\|_1 \leq \tag{116}$$

$$\sum_i\sum_{j;j\neq i}\Bigg\|\int_{\Delta_i}\int_{\Delta_j} dxdyK(x,y)\Gamma(t,x,y)|x\rangle\langle y|\Bigg\|_1 = \tag{117}$$

$$\sum_i\sum_{j;j\neq i}\Bigg\|P_{\Delta_i}\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}_{S_0}\big)P_{\Delta_j}\Bigg\|_1 \tag{118}$$

where $\mathbf{P}_{\Delta_i} := \int_{\Delta_i}|x\rangle\langle x|dx$, i.e. the spectral projector of $\hat{\mathbf{X}}$ projecting onto the subspace corresponding to the set Δ_i.

3.1 Bounds of the Kupsch kind.

Estimating the trace norms in equation (119) below will require us to invoke some ideas from Kupsch's seminal paper on decoherence [38] where it is proven that if Δ_j and Δ_i are intervals with a distance $\delta > 0$, then

$$\|P_{\Delta_i}\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}_{S_0}\big)P_{\Delta_j}\| \leq C(1+\delta^2\psi(t))^{-\gamma} \tag{119}$$

with a function $\psi(t) \geq 0$ which diverges for $t \to \infty$, γ an exponent which can be large and some constant C.

At a first glance, this bound might not seem useful to our work since we are interested in the trace norm rather than the operator norm of the quantity $\hat{\mathbf{P}}_{\Delta_i}\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_0})\hat{\mathbf{P}}_{\Delta_j}$. Furthermore, it is not explicit what γ and $\psi(t)$ should be. We will provide a new version of this result which closely follows the Riemann-Stieltjes integration techniques employed by Kupsch when proving (118) in the appendix of [38].

Theorem 7. *Kupsch like bounds: Let $\hat{\boldsymbol{\rho}}_t$ be some density operator which may be represented using the generalized spectrum of the position operator $\hat{\mathbf{X}}$ as*

$$\hat{\boldsymbol{\rho}}_t = \int\int \Gamma(t,x,y)K_0(x,y)|x\rangle\langle y|dxdy.$$

where $\Gamma(t,x,y) \in C^1(\mathbb{R}^3)$ for all $[0,T)\times\Delta_i\times\Delta_j \subset \mathbb{R}$. We will assume that $\hat{\boldsymbol{\rho}}_0$ is pure and write $\hat{\boldsymbol{\rho}}_0 = |\xi_0\rangle\langle\xi_0|$ in what follows. Furthermore, let $\hat{\mathbf{P}}_\Omega := \int_\Omega dx|x\rangle\langle x|$. Then, for a fixed $t>0$

$$\|\hat{\mathbf{P}}_{\Delta_i}\hat{\boldsymbol{\rho}}_t\hat{\mathbf{P}}_{\Delta_j}\|_1 \le \sup_{(x,y)\in\Delta_i\times\Delta_j}\Big(2|\Gamma(t,x,y)| + |\Delta_j||\partial_y\Gamma(t,x,y)|\Big)$$

when $\Big|\Delta_i\times\Delta_j\bigcap supp\{\Gamma(t,x,y)K_0(x,y)\}\Big| \neq 0$*, otherwise* $\|\hat{\mathbf{P}}_{\Delta_i}\hat{\boldsymbol{\rho}}_t\hat{\mathbf{P}}_{\Delta_j}\|_1 = 0$

Proof. CASE 1)

First consider the set $\Delta_i\times\Delta_j$ with a zero intersection

$$\Big|\Delta_i\times\Delta_j\bigcap supp\{\Gamma(t,x,y)K_0(x,y)\}\Big| = 0$$

In this case

$$\Big\|\hat{\mathbf{P}}_{\Delta_i}\hat{\boldsymbol{\rho}}_t\hat{\mathbf{P}}_{\Delta_j}\Big\|_1 = \Big\|\int_{\Delta_i}\int_{\Delta_j}\Gamma(t,x,y)K_0(x,y)|x\rangle\langle y|dxdy\Big\|_1 =$$

$$\Big\|\int_{\Delta_i}\int_{\Delta_j}0|x\rangle\langle y|dxdy\Big\|_1 = 0\|\hat{\mathbf{P}}_{\Delta_i}\hat{\mathbf{P}}_{\Delta_j}\|_1 = 0.$$

CASE 2)

Now assume that

$$\Big|\Delta_i\times\Delta_j\bigcap supp\{\Gamma(t,x,y)K_0(x,y)\}\Big| \neq 0$$

Let us begin by considering the operator

$$\hat{\mathbf{T}}_t(y) := \int_{\Delta_i}\Gamma(t,x,y)|x\rangle\langle x|dx.$$

Where i is fixed. $\hat{T}_t(y)$ is a differentiable family of operators, with respect to y, with the operator norm estimates

$$\|\hat{\mathbf{T}}_t(y)\| \le \sup_{x\in\Delta_i}|\Gamma(t,x,y)|$$

since

$$\|\hat{\mathbf{T}}_t(y)\|^2 = \sup_{\||\psi\rangle\|=1}\|\hat{\mathbf{T}}_t(y)|\psi\rangle\|^2 = \sup_{\||\psi\rangle\|=1}\int_{\Delta_i}\int_{\Delta_i}\Gamma(t,x',y)^*\Gamma(t,x,y)\langle\psi|x'\rangle\langle x'|x\rangle\langle x|\psi\rangle dx'dx =$$

$$\sup_{\||\psi\rangle\|=1}\int_{\Delta_i}|\Gamma(t,x,y)|^2\langle\psi|x\rangle\langle x|\psi\rangle dx \le \sup_{x\in\Delta_i}|\Gamma(t,x,y)|^2\sup_{\||\psi\rangle\|=1}\int_{\Delta_i}|\psi(x)|^2dx \le$$

$$\sup_{x\in\Delta_i}|\Gamma(t,x,y)|^2$$

In a similar way we may bound the operator $\hat{\mathbf{T}}'_t(y) := \int_{\Delta_i}\Gamma^{'}(t,x,y)|x\rangle|\langle x|dx$. Where $\Gamma^{'}(t,x,y) :=$

$\partial_y\Gamma(t,x,y)$.i.e.

$$\|\hat{\mathbf{T}}'_t(y)\| \le \sup_{x\in\Delta_i}|\Gamma^{'}(t,x,y)|$$

Furthermore, define $\hat{\mathbf{J}}_t(y) := \hat{\mathbf{T}}_t(y)\hat{\boldsymbol{\rho}}_0$ and $\hat{\mathbf{J}}_t^{'}(y) := \hat{\mathbf{T}}_t^{'}(y)\hat{\boldsymbol{\rho}}_0$. These operators also have uniform estimates; due to the estimates computed above, and the inequality $\|AC\|_1 \leq \|A\|\|B\|_1$ one may easily show that

$$\|\hat{\mathbf{J}}_t(y)\|_1 \leq \sup_{x\in\Delta_i} |\Gamma(t,x,y)|\left\|\hat{\rho}_0\right\|_1 = \sup_{x\in\Delta_i} |\Gamma(t,x,y)|$$

and that

$$\|\hat{\mathbf{J}}_t^{'}(y)\|_1 \leq \sup_{x\in\Delta_i} |\Gamma^{'}(t,x,y)|\left\|\hat{\rho}_0\right\|_1 = \sup_{x\in\Delta_i} |\Gamma^{'}(t,x,y)|.$$

Before we proceed we show the relationship between the operator $\hat{\mathbf{T}}_t'(y)$ and the weak derivative $\partial_y\langle\psi|\hat{\mathbf{T}}_t(y)|\phi\rangle$.

$$\partial_y\langle\psi|\hat{\mathbf{T}}_t(y)|\phi\rangle = \partial_y\int_{\Delta_i}\Gamma(t,x,y)\langle\psi|x\rangle\langle x|\phi\rangle dx$$

We assumed that $\Gamma(t,x,y)$ is $C^1(\Delta_i)$ in both x and y for any Δ_i, therefore we may swap the order of the integral and the derivative.

$$\partial_y\int_{\Delta_i}\Gamma(t,x,y)\langle\psi|x\rangle\langle x|\phi\rangle dx = \int_{\Delta_i}\partial_y\Gamma(t,x,y)\langle\psi|x\rangle\langle x|\phi\rangle dx =$$

$$\int_{\Delta_i}\Gamma^{'}(t,x,y)\langle\psi|x\rangle\langle x|\phi\rangle dx = \langle\psi|\Big(\int_{\Delta_i}\Gamma^{'}(t,x,y)|x\rangle\langle x|dx\Big)|\phi\rangle = \langle\psi|\hat{T}_t'(y)|\phi\rangle$$

We therefore have

$$\partial_y\langle\psi|\hat{\mathbf{T}}_t(y)|\phi\rangle = \langle\psi|\hat{\mathbf{T}}_t'(y)|\phi\rangle \tag{120}$$

Now, for all intervals Δ_j we have $\int_{\Delta_j}\hat{\mathbf{J}}_t(y)|y\rangle\langle y|dy = \hat{\mathbf{P}}_{\Delta_i}\hat{\boldsymbol{\rho}}_t\hat{\mathbf{P}}_{\Delta_j}$. We write $\Delta_j := [a_j, b_j]$. We will show that the following identity holds.

$$\int_{\Delta_j}\hat{\mathbf{J}}_t(y)|y\rangle\langle y|dy = \hat{\mathbf{J}}_t(b_j)\hat{\mathbf{P}}_{(-\infty,b_j]} - \hat{\mathbf{J}}_t(a_j)\hat{\mathbf{P}}_{(-\infty,a_j]} - \int_{\Delta_j}\hat{\mathbf{J}}_t'(y)\hat{\mathbf{P}}_{(-\infty,y]}dy. \tag{121}$$

For arbitrary $|\psi\rangle$ and $|\phi\rangle$

$$\langle\psi|\int_{\Delta_j}\hat{\mathbf{J}}_t(y)|y\rangle\langle y|dy|\,\phi\rangle = \int_{\Delta_j}\langle\psi|\hat{\mathbf{J}}_t(y)|y\rangle\langle y|\phi\rangle dy. \tag{122}$$

By the definition of $\hat{\mathbf{J}}_t(y)$ one has

$$\langle\psi|\hat{\mathbf{J}}_t(y)|y\rangle = \langle\psi|\hat{\mathbf{T}}_t(y)\hat{\rho}_0|y\rangle = \langle\psi|\hat{\mathbf{T}}_t(y)|\xi_0\rangle\langle\xi_0|y\rangle. \tag{123}$$

Picking up from (122).

$$(122) = \int_{\Delta_j}\langle\psi|\hat{\mathbf{T}}_t(y)|\xi_0\rangle\langle\xi_0|y\rangle\langle y|\phi\rangle dy = \tag{124}$$

$$\left[\langle\psi|\hat{\mathbf{T}}_t(y)|\xi_0\rangle\langle\xi_0|\Big(\int_{-\infty}^{y}dy'|y'\rangle\langle y'|\Big)|\phi\rangle\right]\Bigg|_{a_j}^{b_j} - \int_{\Delta_j}\Big(\int_{-\infty}^{y}\langle\xi_0|y'\rangle\langle y'|\phi\rangle dy'\Big)d\Big(\langle\psi|\hat{\mathbf{T}}_t(y)|\xi_0\rangle\Big) = \tag{125}$$

$$\left[\langle\psi|\hat{\mathbf{T}}_t(y)|\xi_0\rangle\langle\xi_0|P_{(-\infty,y]}|\phi\rangle\right]\Bigg|_{a_j}^{b_j} - \int_{\Delta_j}\Big(\int_{-\infty}^{y}\langle\xi_0|y'\rangle\langle y'|\phi\rangle dy'\Big)\Big(\langle\psi|\hat{\mathbf{T}}_t(y)|\xi_0\rangle\Big)'dy = \tag{126}$$

$$\left[\langle\psi|\hat{\mathbf{T}}_t(y)|\xi_0\rangle\langle\xi_0|\hat{\mathbf{P}}_{(-\infty,y]}|\phi\rangle\right]\Bigg|_{a_j}^{b_j} - \int_{\Delta_j}\Big(\langle\psi|\hat{\mathbf{T}}_t(y)|\xi_0\rangle\Big)'\langle\xi_0|\hat{\mathbf{P}}_{(-\infty,y]}|\phi\rangle dy = \tag{127}$$

$$\langle\psi|\left[\hat{\mathbf{T}}_t(y)|\xi_0\rangle\langle\xi_0|\hat{\mathbf{P}}_{(-\infty,y]}\right]\Bigg|_{a_j}^{b_j}|\phi\rangle - \langle\psi|\int_{\Delta_j}\hat{\mathbf{T}}'_t(y)|\xi_0\rangle\langle\xi_0|\hat{\mathbf{P}}_{(-\infty,y]}dy|\phi\rangle = \tag{128}$$

$$\langle\psi|\left(\hat{\mathbf{J}}_t(y)\hat{\mathbf{P}}_{(-\infty,y]}\Bigg|_{a_j}^{b_j} - \int_{\Delta_j}\hat{\mathbf{J}}'_t(y)\hat{\mathbf{P}}_{(-\infty,y]}dy\right)|\phi\rangle = \tag{129}$$

$$\langle\psi|\left(\hat{\mathbf{J}}_t(b_j)\hat{\mathbf{P}}_{(-\infty,b_j]} - \hat{J}_t(a_j)\hat{\mathbf{P}}_{(-\infty,a_j]} - \int_{\Delta_j}\hat{\mathbf{J}}'_t(y)\hat{\mathbf{P}}_{(-\infty,y]}dy\right)|\phi\rangle = \tag{130}$$

We therefore have

$$\|\hat{\mathbf{P}}_{\Delta_i}\hat{\boldsymbol{\rho}}_t\hat{\mathbf{P}}_{\Delta_j}\|_1 = \left\|\int_{\Delta_j}\hat{\mathbf{J}}_t(y)|y\rangle\langle y|dy\right\|_1 =$$
$$\left\|\hat{\mathbf{J}}_t(b_j)\hat{\mathbf{P}}_{(-\infty,b_j]} - \hat{\mathbf{J}}_t(a_j)\hat{\mathbf{P}}_{(-\infty,a_j]} - \int_{\Delta_j}\hat{\mathbf{J}}'_t(y)\hat{\mathbf{P}}_{(-\infty,y]}dy\right\|_1 \leq$$
$$\|\hat{\mathbf{J}}_t(b_j)\hat{\mathbf{P}}_{(-\infty,b_j]}\|_1 + \|\hat{\mathbf{J}}_t(a_j)\hat{\mathbf{P}}_{(-\infty,a_j]}\|_1 + \left\|\int_{\Delta_j}\hat{\mathbf{J}}'_t(y)\hat{\mathbf{P}}_{(-\infty,y]}dy\right\|_1 \leq$$
$$\|\hat{\mathbf{J}}_t(b_j)\|_1\|\hat{\mathbf{P}}_{(-\infty,b_j]}\| + \|\hat{\mathbf{J}}_t(a_j)\|_1\|\hat{\mathbf{P}}_{(-\infty,a_j]}\| + \int_{\Delta_j}\|\hat{\mathbf{J}}_t(y)'\hat{\mathbf{P}}_{(-\infty,y]}\|_1 dy \leq$$
$$\|\hat{\mathbf{J}}_t(b_j)\|_1 + \|\hat{\mathbf{J}}_t(a_j)\|_1 + \int_{\Delta_j}\|\hat{\mathbf{J}}'_t(y)\|_1\|\hat{\mathbf{P}}_{(-\infty,y]}\|dy =$$
$$\|\hat{\mathbf{J}}_t(b_j)\|_1 + \|\hat{\mathbf{J}}_t(a_j)\|_1 + \int_{\Delta_j}\|\hat{\mathbf{J}}'_t(y)\|_1 dy \leq$$
$$\|\hat{\mathbf{J}}_t(b_j)\|_1 + \|\hat{\mathbf{J}}_t(a_j)\|_1 + |\Delta_j|\sup_{y\in\Delta_j}\|\hat{\mathbf{J}}'_t(y)\|_1 dy \leq$$
$$\sup_{x\in\Delta_i}|\Gamma(t,x,b_j)| + \sup_{x\in\Delta_i}|\Gamma(t,x,a_j)| + |\Delta_j|\sup_{x\in\Delta_i, y\in\Delta_j}|\partial_y\Gamma(t,x,y)| \leq$$
$$\sup_{(x,y)\in\Delta_i\times\Delta_j}\left(2|\Gamma(t,x,y)| + |\Delta_j||\Gamma^{'}(t,x,y)|\right)$$

□

3.2 Estimating the diagonal term

We have hitherto developed the tools necessary to estimate the trace norm of the "off-diagonal" terms of the density operator (112). We shall now study the "diagonal" terms of the same state (112). i.e.

$$\hat{\boldsymbol{\rho}}_t = \sum_i \int_{\Delta_i}\int_{\Delta_i} dxdyK(x,y)\Gamma(t,x,y)|x\rangle\langle y| \otimes \hat{\boldsymbol{\rho}}_{x,y}^{E_t^1}. \tag{131}$$

For the "off-diagonal" terms we were simply interested in bounding the totality of the terms in order to estimate the asymptotic behavior. For the diagonal terms we are interested in the asymptotic limit of the following trace norm optimization.

$$\min_{PVM}\left\|\sum_i\left(\hat{\mathbf{P}}_{\Delta_i}\otimes\mathbb{I}\right)\hat{\boldsymbol{\rho}}_t\left(\hat{\mathbf{P}}_{\Delta_i}\otimes\mathbb{I}\right) - \sum_i\left(\hat{\mathbf{P}}_{\Delta_i}\otimes\hat{\mathbf{P}}_i^{E_t^1}\right)\hat{\boldsymbol{\rho}}_t\left(\hat{\mathbf{P}}_{\Delta_i}\otimes\hat{\mathbf{P}}_i^{E_t^1}\right)\right\|_1 \tag{132}$$

The minimization is taken over all PVMs (projection valued measures) resolving the identity operator of the space associated with the environmental degrees of freedom. The term $\sum_i\left(\hat{\mathbf{P}}_{\Delta_i}\otimes\mathbb{I}\right)\hat{\boldsymbol{\rho}}_t\left(\hat{\mathbf{P}}_{\Delta_i}\otimes\mathbb{I}\right)$ is indeed just another way o writing the state (131). The use of the PVM $\{\hat{\mathbf{P}}_{\Delta_i}\}_i$ in the first term of (132) found in (132) is just technical. However, the usage of the same PVM on the second term in the difference of (132) does imply measurement of the von Neumann type performed on the system; i.e in a local sense in the sense of the following definition.

Definition 4. *Unread von Neumann measurement: Consider a PVM* $\{\hat{\mathbf{P}}_i\}_i$ *acting in some* Hilbert *space of arbitrary dimension. Furthermore, consider a density operator* $\hat{\rho}$ *which acts in the same* Hilbert *space. Each element of the PVM is a projection onto an eigenspace of the self-adjoint operator modeling the measurement apparatus. The density operator of the system after obtaining the measurement result* i *is*

$$\frac{\hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}\hat{\mathbf{P}}_i}{Tr\{\hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}\hat{\mathbf{P}}_i\}}$$

The state above is the resulting state assuming that we have "read out" the measurement. However, if we do not read out the results of the measurement what we have is a mixture

$$\sum_i p_i \frac{\hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}\hat{\mathbf{P}}_i}{Tr\{\hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}\hat{\mathbf{P}}_i\}}$$

where p_i *is the probability of the* i*th outcome of the measurement. Since* $p_i = Tr\{\hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}\}$, *the unread state of the system is*

$$\sum_i p_i \frac{\hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}\hat{\mathbf{P}}_i}{Tr\{\hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}\hat{\mathbf{P}}_i\}} = \sum_i Tr\{\hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}\hat{\mathbf{P}}_i\} \frac{\hat{\mathbf{P}}_i\hat{\rho}\hat{\mathbf{P}}_i}{Tr\{\hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}\hat{\mathbf{P}}_i\}} = \sum_i \hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}\hat{\mathbf{P}}_i$$

One might have already noted that the map $\sum_i \hat{\mathbf{P}}_{\Delta_i} \otimes \hat{\mathbf{P}}_i^{E_t^1}(...)\hat{\mathbf{P}}_{\Delta_i} \otimes \hat{\mathbf{P}}_i^{E_t^1}$ is unlike the related measurements of von Neumann type seen in definition (4) since they do not preserve the trace of a density matrix. Both are indeed completely positive maps but the latter map turns out to reduce the trace in general, i.e.

$$Tr\left\{\sum_i \left(\hat{\mathbf{P}}_{\Delta_i} \otimes \hat{\mathbf{P}}_i^{E_t^1}\right)\hat{\boldsymbol{\rho}}_t\left(\hat{\mathbf{P}}_{\Delta_i} \otimes \hat{\mathbf{P}}_i^{E_t^1}\right)\right\} \leq Tr\{\hat{\boldsymbol{\rho}}_t\} = 1. \tag{133}$$

Indeed the PVM $\{\hat{\mathbf{P}}_{\Delta_i} \otimes \hat{\mathbf{P}}_i^{E_t^1}\}_i$ by itself does not describe a measurement for the product of the systems and environments Hilbert spaces because it does not resolve the identity. The associated PVM will indeed be the family of projectors $\{\hat{\mathbf{P}}_{\Delta_i} \otimes \hat{\mathbf{P}}_j^{E_t^1}\}_{i,j}$. i.e. including situations where the environment measures an outcome j different from the outcome measured by the system $i \neq j$.

Let us now estimate (132). We rewrite the operator $\sum_i \left(\hat{\mathbf{P}}_{\Delta_i} \otimes \mathbb{I}\right)\hat{\boldsymbol{\rho}}_t\left(\hat{\mathbf{P}}_{\Delta_i} \otimes \mathbb{I}\right)$ in the form:

$$\sum_i \left(\hat{\mathbf{P}}_{\Delta_i} \otimes \mathbb{I}\right)\hat{\boldsymbol{\rho}}_t\left(\hat{\mathbf{P}}_{\Delta_i} \otimes \mathbb{I}\right) = \sum_i \int_{\Delta_i}\int_{\Delta_i} dxdy K(x,y)\Gamma(t,x,y)|x\rangle\langle y| \otimes \hat{\boldsymbol{\rho}}_{x,y}^{E_t^1} = \tag{134}$$

$$\sum_i \bar{p}_i \int_{\Delta_i}\int_{\Delta_i} dxdy \frac{K(x,y)}{\bar{p}_i}\Gamma(t,x,y)|x\rangle\langle y| \otimes \hat{\boldsymbol{\rho}}_{x,y}^{E_t^1} = \tag{135}$$

where

$$\bar{p}_i := \int_{\Delta_i} K(x,x)dx.$$

That is,

$$(135) = \sum_i \bar{p}_i \int_{\mathbb{R}}\int_{\mathbb{R}} K_i(x,y)\Gamma(t,x,y)|x\rangle\langle y| \otimes \hat{\boldsymbol{\rho}}_{x,y}^{E_t^1} \tag{136}$$

where we now define

$$K_i(x,y) := \mathbb{1}_{\Delta_i}(x)\mathbb{1}_{\Delta_i}(y)\frac{K(x,y)}{\bar{p}_i} = \frac{\mathbb{1}_{\Delta_i}(x)\psi(x)}{\sqrt{\bar{p}_i}}\frac{\mathbb{1}_{\Delta_i}(y)\psi^*(y)}{\sqrt{\bar{p}_i}}$$

recalling that

$$K(x,y) = \psi(x)\psi^*(y).$$

since the initial state of the system is pure. Furthermore, let us define

$$\psi_{S_i}(x) := \frac{\mathbb{1}_{\Delta_i}(x)\psi(x)}{\sqrt{\bar{p}_i}}$$

and write

$$K_i(x,y) = \psi^*_{S_i}(x)\psi_{S_i}(y).$$

Finally,

$$(136) = \sum_i \bar{p}_i \mathscr{U}_{1,t}\big(\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_i}) \otimes \hat{\boldsymbol{\rho}}^{E_0^1}\big) \tag{137}$$

defining

$$\hat{\boldsymbol{\rho}}_{S_i} := \frac{\hat{\mathbf{P}}_{\Delta_i}\hat{\boldsymbol{\rho}}_{S_0}\hat{\mathbf{P}}_{\Delta_i}}{Tr\big\{\hat{\mathbf{P}}_{\Delta_i}\hat{\boldsymbol{\rho}}_{S_0}\hat{\mathbf{P}}_{\Delta_i}\big\}} = \int\int K_i(x,y)|x\rangle\langle y|dxdy \tag{138}$$

and reminding the reader that

$$\mathscr{U}_{1,t}(\hat{\mathbf{A}}) := e^{-it\gamma f(\hat{\mathbf{X}})\otimes g(\hat{\mathbf{B}})}\hat{\mathbf{A}}e^{it\gamma f(\hat{\mathbf{X}})\otimes g(\hat{\mathbf{B}})}.$$

We may now write

$$\sum_i \Big(\hat{\mathbf{P}}_{\Delta_i}\otimes\hat{\mathbf{P}}_i^{E_t^1}\Big)\hat{\boldsymbol{\rho}}_t\Big(\hat{\mathbf{P}}_{\Delta_i}\otimes\hat{\mathbf{P}}_i^{E_t^1}\Big) = \sum_i \bar{p}_i\Big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\Big)\mathscr{U}_{1,t}\big(\mathscr{E}_t^1(\hat{\boldsymbol{\rho}}_{S_i})\otimes\hat{\boldsymbol{\rho}}^{E_0}\big)\Big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\Big) \tag{139}$$

Finally, normalizing the operator (139) we obtain

$$\hat{\boldsymbol{\rho}}_{CSBS,t} := \sum_i \bar{p}_i\Big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\Big)\mathscr{U}_{1,t}\big(\mathscr{E}_t^1(\hat{\boldsymbol{\rho}}_{S_i})\otimes\hat{\boldsymbol{\rho}}^{E_0}\big)\Big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\Big). \tag{140}$$

The notation CSBS stands for continuous variables SBS, this is a concept that will generalize the concept of SBS to the CV case. A more in-depth discussion regarding the continuous variables SBS will be had in following sections. As for the discrete variables case, we will be estimating $\|\hat{\boldsymbol{\rho}}_t - \mathscr{N}\hat{\boldsymbol{\rho}}_{CSBS,t}\|_1$ and then using Lemma 1 to bound $\|\hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{CSBS,t}\|_1$.

Utilizing this new representation for (134) and the left-hand side of the equation (139), we can see more clearly that the quantum map $\mathscr{E}_t$ may be avoided by exploiting the inequality $\|\mathscr{M}_t(\hat{\boldsymbol{\sigma}})\|_1 \leq \|\hat{\boldsymbol{\sigma}}\|_1$, known to be true for all density operators $\hat{\sigma}$ and quantum maps $\mathscr{M}_t$ and the following theorem.

Theorem 8. *Let $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\eta}}$ all be pure density operators acting respectively in the Hilbert spaces $\mathscr{H}_S$, $\mathscr{H}_{E_1}$, and $\mathscr{H}_{E_2}$. We let $\hat{\mathbf{X}}$ be a position operator. Furthermore, let $\hat{\mathbf{X}}$, $\hat{\mathbf{B}}_1$ and $\hat{\mathbf{B}}_2$ be either position or momentum operators acting respectively in the Hilbert spaces $\mathscr{H}_S$, $\mathscr{H}_{E_1}$, and $\mathscr{H}_{E_2}$. Let $\hat{\mathbf{P}}$ be a bounded operator acting in the Hilbert space $\mathscr{H}_{E_1}$. Finally, assume that $\hat{\mathbf{A}}$ is a trace class operator and let us define*

$$\mathscr{U}_t(\hat{\mathbf{A}}) := e^{-it\hat{\mathbf{X}}\otimes\hat{\mathbf{B}}_1}\hat{\mathbf{A}}e^{it\hat{\mathbf{X}}\otimes\hat{\mathbf{B}}_1},$$

$$\mathscr{E}_t(\hat{\mathbf{A}}) := Tr_{E_2}\Big\{\Big(e^{-it\hat{\mathbf{X}}\otimes\mathbb{I}_{E_1}\otimes\hat{\mathbf{B}}_2}\Big)\Big(\hat{\mathbf{A}}\otimes\hat{\boldsymbol{\eta}}\Big)\Big(e^{it\hat{\mathbf{X}}\otimes\mathbb{I}_{E_1}\otimes\hat{\mathbf{B}}_2}\Big)\Big\},$$

and

$$\mathscr{E}_{S,t}(\hat{\mathbf{A}}) := \int\int\langle x|\hat{\mathbf{A}}|y\rangle Tr_{E_2}\Big\{e^{-itx\hat{\mathbf{B}}_2}\hat{\boldsymbol{\eta}}e^{ity\hat{\mathbf{B}}_2}\Big\}|x\rangle\langle y|dxdy$$

then

$$\mathscr{E}_t\Big(\Big(\mathbb{I}_S\otimes\hat{\mathbf{P}}\Big)\mathscr{U}_t(\hat{\boldsymbol{\rho}}\otimes\hat{\boldsymbol{\sigma}})\Big(\mathbb{I}_S\otimes\hat{\mathbf{P}}\Big)\Big) = \Big(\mathbb{I}_S\otimes\hat{\mathbf{P}}\Big)\mathscr{U}_t\Big(\mathscr{E}_{S,t}(\hat{\boldsymbol{\rho}})\otimes\hat{\boldsymbol{\sigma}}\Big)\Big(\mathbb{I}_S\otimes\hat{\mathbf{P}}\Big)$$

Proof.

$$\mathscr{E}_t\Big(\Big(\mathbb{I}_S\otimes\hat{\mathbf{P}}\Big)\mathscr{U}_t(\hat{\boldsymbol{\rho}}\otimes\hat{\boldsymbol{\sigma}})\Big(\mathbb{I}_S\otimes\hat{\mathbf{P}}\Big)\Big) = \tag{141}$$

$$Tr_{E_2}\Big\{\Big(e^{-it\hat{\mathbf{X}}\otimes\mathbb{I}_{E_1}\otimes\hat{\mathbf{B}}_2}\Big)\Big(\Big(\mathbb{I}_S\otimes\hat{\mathbf{P}}\Big)\mathscr{U}_t\Big(\mathscr{E}_{S,t}(\hat{\boldsymbol{\rho}})\otimes\hat{\boldsymbol{\sigma}}\Big)\Big(\mathbb{I}_S\otimes\hat{\mathbf{P}}\Big)\otimes\hat{\boldsymbol{\eta}}\Big)\Big(e^{it\hat{\mathbf{X}}\otimes\mathbb{I}_{E_1}\otimes\hat{\mathbf{B}}_2}\Big)\Big\} = \tag{142}$$

$$Tr_{E_2}\Big\{\Big(e^{-it\hat{\mathbf{X}}\otimes\mathbb{I}_{E_1}\otimes\hat{\mathbf{B}}_2}\Big)\Big(\Big(\mathbb{I}_S\otimes\hat{\mathbf{P}}\Big)\Big(\int\int K(x,y)|x\rangle\langle y|\otimes e^{-tx\hat{\mathbf{B}}_1}\hat{\boldsymbol{\sigma}}e^{ity\hat{\mathbf{B}}_1}dxdy\Big)\Big(\mathbb{I}_S\otimes\hat{\mathbf{P}}\Big)\otimes\hat{\boldsymbol{\eta}}\Big)\Big(e^{it\hat{\mathbf{X}}\otimes\mathbb{I}_{E_1}\otimes\hat{\mathbf{B}}_2}\Big)\Big\}= \tag{143}$$

$$Tr_{E_2}\Big\{\Big(e^{-it\hat{\mathbf{X}}\otimes\mathbb{I}_{E_1}\otimes\hat{\mathbf{B}}_2}\Big)\Big(\Big(\int\int K(x,y)|x\rangle\langle y|\otimes\hat{\mathbf{P}}e^{-tx\hat{\mathbf{B}}_1}\hat{\boldsymbol{\sigma}}e^{ity\hat{\mathbf{B}}_1}\hat{\mathbf{P}}dxdy\Big)\otimes\hat{\boldsymbol{\eta}}\Big)\Big(e^{it\hat{\mathbf{X}}\otimes\mathbb{I}_{E_1}\otimes\hat{\mathbf{B}}_2}\Big)\Big\}= \tag{144}$$

$$Tr_{E_2}\Big\{\int\int K(x,y)|x\rangle\langle y|\otimes\hat{\mathbf{P}}e^{-tx\hat{\mathbf{B}}_1}\hat{\boldsymbol{\sigma}}e^{ity\hat{\mathbf{B}}_1}\hat{\mathbf{P}}\otimes e^{-itx\hat{\mathbf{B}}_2}\hat{\boldsymbol{\eta}}e^{ity\hat{\mathbf{B}}_2}dxdy\Big\}= \tag{145}$$

$$\int\int K(x,y)Tr_{E_2}\Big\{e^{-itx\hat{\mathbf{B}}_2}\hat{\boldsymbol{\eta}}e^{ity\hat{\mathbf{B}}_2}\Big\}|x\rangle\langle y|\otimes\hat{\mathbf{P}}e^{-tx\hat{\mathbf{B}}_1}\hat{\boldsymbol{\sigma}}e^{ity\hat{\mathbf{B}}_1}\hat{\mathbf{P}}dxdy= \tag{146}$$

$$\hat{\mathbf{P}}e^{-it\hat{\mathbf{X}}\otimes\hat{\mathbf{B}}_1}\Big(\int\int K(x,y)Tr_{E_2}\Big\{e^{-itx\hat{\mathbf{B}}_2}\hat{\boldsymbol{\eta}}e^{ity\hat{\mathbf{B}}_2}\Big\}|x\rangle\langle y|\otimes\hat{\boldsymbol{\sigma}}dxdy\Big)e^{it\hat{\mathbf{X}}\otimes\hat{\mathbf{B}}_1}\hat{\mathbf{P}}= \tag{147}$$

$$\Big(\mathbb{I}_S\otimes\hat{\mathbf{P}}\Big)e^{-it\hat{\mathbf{X}}\otimes\hat{\mathbf{B}}_1}\Big(\int\int K(x,y)Tr_{E_2}\Big\{e^{-itx\hat{\mathbf{B}}_2}\hat{\boldsymbol{\eta}}e^{ity\hat{\mathbf{B}}_2}\Big\}|x\rangle\langle y|\otimes\hat{\boldsymbol{\sigma}}dxdy\Big)e^{it\hat{\mathbf{X}}\otimes\hat{\mathbf{B}}_1}\Big(\mathbb{I}\otimes\hat{\mathbf{P}}\Big)= \tag{148}$$

$$\Big(\mathbb{I}_S\otimes\hat{\mathbf{P}}\Big)e^{-it\hat{\mathbf{X}}\otimes\hat{\mathbf{B}}_1}\Big(\mathscr{E}_{S,t}\big(\hat{\boldsymbol{\rho}}\big)\otimes\hat{\boldsymbol{\sigma}}\Big)e^{it\hat{\mathbf{X}}\otimes\hat{\mathbf{B}}_1}\Big(\mathbb{I}\otimes\hat{\mathbf{P}}\Big) \tag{149}$$

□

Now we estimate the following trace distance.

$$\Big\|\sum_i\Big(\hat{\mathbf{P}}_{\Delta_i}\otimes\mathbb{I}\Big)\hat{\boldsymbol{\rho}}_t\Big(\hat{\mathbf{P}}_{\Delta_i}\otimes\mathbb{I}\Big)-\sum_i\Big(\hat{\mathbf{P}}_{\Delta_i}\otimes\hat{\mathbf{P}}_i^{E_t^1}\Big)\hat{\boldsymbol{\rho}}_t\Big(\hat{\mathbf{P}}_{\Delta_i}\otimes\hat{\mathbf{P}}_i^{E_t^1}\Big)\Big\|_1= \tag{150}$$

$$\Big\|\sum_i\bar{p}_i\Big(\mathscr{U}_{1,t}\big(\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}_{S_i}\big)\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)-\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)\mathscr{U}_{1,t}\big(\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}_{S_i}\big)\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)\Big)\Big\|_1\leq \tag{151}$$

$$\sum_i\bar{p}_i\Big\|\mathscr{U}_{1,t}\big(\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}_{S_i}\big)\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)-\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)\mathscr{U}_{1,t}\big(\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}_{S_i}\big)\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)\Big\|_1\leq \tag{152}$$

$$\sum_i\bar{p}_i\Big\|\mathscr{E}_t\Big(\mathscr{U}_{1,t}\big(\hat{\boldsymbol{\rho}}_{S_i}\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)-\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)\mathscr{U}_{1,t}\big(\hat{\boldsymbol{\rho}}_{S_i}\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)\Big)\Big\|_1\leq \tag{153}$$

$$\sum_i\bar{p}_i\Big\|\mathscr{U}_{1,t}\big(\hat{\boldsymbol{\rho}}_{S_i}\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)-\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)\mathscr{U}_{1,t}\big(\hat{\boldsymbol{\rho}}_{S_i}\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)\Big\|_1 \tag{154}$$

where we have used Theorem 8 going from (152) to (153) in order to reorder the composition of the maps present. Without the effects of the quantum map $\mathscr{E}_t$ the local kernels of the system and environmental degrees of freedom are henceforth separable. In fact, the state $\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_i}\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)$ above is pure! To accentuate the latter we write said state as follows.

$$\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_i}\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big):=\hat{\mathbf{U}}_t\Big(|\psi_{S_i}\rangle\langle\psi_{S_i}|\otimes|\psi_{E_0^1}\rangle\langle\psi_{E_0^1}|\Big)\hat{\mathbf{U}}_t^\dagger=\Big(\hat{\mathbf{U}}_t\Big(|\psi_{S_i}\rangle\otimes|\psi_{E_0^1}\rangle\Big)\Big)\Big(\hat{\mathbf{U}}_t\Big(|\psi_{S_i}\rangle\otimes|\psi_{E_0^1}\rangle\Big)\Big)^\dagger. \tag{155}$$

from which it follows that

$$\Big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\Big)\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_i}\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)\Big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\Big)=\Big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\hat{\mathbf{U}}_t\Big(|\psi_{S_i}\rangle\otimes|\psi_{E_0^1}\rangle\Big)\Big)\Big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\hat{\mathbf{U}}_t\Big(|\psi_{S_i}\rangle\otimes|\psi_{E_0^1}\rangle\Big)\Big)^\dagger. \tag{156}$$

where of course $\hat{\mathbf{U}}_t:=e^{-it\gamma f(\hat{\mathbf{X}})\otimes g(\hat{\mathbf{B}})}$.

Before continuing we will define the following significant object.

$$\mathscr{N}_i:=Tr\Big\{\Big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\Big)\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_i}\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)\Big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\Big)\Big\}= \tag{157}$$

$$\langle\psi_{S_i}|\otimes\langle\psi_{E_0^1}|\hat{\mathbf{U}}_t^\dagger\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)\hat{\mathbf{U}}_t|\psi_{S_i}\rangle\otimes|\psi_{E_0^1}\rangle = \tag{158}$$

$$\int dy\psi^*_{S_i}(y)\langle y|\otimes\langle\psi_{E_0^1}|\hat{\mathbf{U}}_t^\dagger\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)\hat{\mathbf{U}}_t\int dx\psi_{S_i}(x)|x\rangle\otimes|\psi_{E_0^1}\rangle = \tag{159}$$

$$\int\int dydxK_i(x,y)\langle y|x\rangle\langle\psi_{E_0^1}|e^{it\gamma f(y)g(\hat{\mathbf{B}})}\hat{\mathbf{P}}_i^{E_t^1}e^{-it\gamma f(x)g(\hat{\mathbf{B}})}|\psi_{E_0^1}\rangle = \tag{160}$$

$$\int dx|\psi_{S_i}(x)|^2\langle\psi_{E_0^1}|e^{it\gamma f(x)g(\hat{\mathbf{B}})}\hat{\mathbf{P}}_i^{E_t^1}e^{-it\gamma f(x)g(\hat{\mathbf{B}})}|\psi_{E_0^1}\rangle = \tag{161}$$

$$\int|\psi_{S_i}(x)|^2\langle\psi_{E_0^1}|e^{it\gamma f(x)g(\hat{\mathbf{B}})}\hat{\mathbf{P}}_i^{E_t^1}\hat{\mathbf{P}}_i^{E_t^1}e^{-it\gamma f(x)g(\hat{\mathbf{B}})}|\psi_{E_0^1}\rangle = \tag{162}$$

$$\int dx|\psi_{S_i}(x)|^2Tr\big\{\hat{\mathbf{P}}_i^{E_t^1}e^{-it\gamma f(x)g(\hat{\mathbf{B}})}|\psi_{E_0^1}\rangle\langle\psi_{E_0^1}|e^{it\gamma f(x)g(\hat{\mathbf{B}})}\hat{\mathbf{P}}_i^{E_t^1}\big\} = \tag{163}$$

$$\int dx|\psi_{S_i}(x)|^2Tr\big\{\hat{\mathbf{P}}_i^{E_t^1}e^{-it\gamma f(x)g(\hat{\mathbf{B}})}\hat{\boldsymbol{\rho}}^{E_0^1}e^{it\gamma f(x)g(\hat{\mathbf{B}})}\hat{\mathbf{P}}_i^{E_t^1}\big\} = \tag{164}$$

$$\int dx|\psi_{S_i}(x)|^2Tr\big\{\hat{\mathbf{P}}_i^{E_t^1}\hat{\boldsymbol{\rho}}_{x,x}^{E_t^1}\hat{\mathbf{P}}_i^{E_t^1}\big\} = \tag{165}$$

$$Tr\bigg\{\hat{\mathbf{P}}_i^{E_t^1}\bigg(\int dx|\psi_{S_i}(x)|^2\hat{\boldsymbol{\rho}}_{x,x}^{E_t^1}\bigg)\hat{\mathbf{P}}_i^{E_t^1}\bigg\} := \tag{166}$$

$$Tr\bigg\{\hat{\mathbf{P}}_i^{E_t^1}\Lambda_{t,i}\big(\hat{\boldsymbol{\rho}}^{E_0^1}\big)\hat{\mathbf{P}}_i^{E_t^1}\bigg\} \tag{167}$$

where we define the quantum map $\Lambda_{t,i}$ as follows.

$$\Lambda_{t,i}\big(\hat{\boldsymbol{\rho}}\big) := \int dx|\psi_{S_i}(x)|^2e^{-it\gamma f(x)g(\hat{\mathbf{B}})}\hat{\boldsymbol{\rho}}e^{it\gamma f(x)g(\hat{\mathbf{B}})}. \tag{168}$$

We are now ready to estimate (154).

$$\sum_i\bar{p}_i\Big\|\mathscr{U}_{1,t}\big(\hat{\boldsymbol{\rho}}_{S_i}\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)-\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)\mathscr{U}_{1,t}\big(\hat{\boldsymbol{\rho}}_{S_i}\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)\Big\|_1 = \tag{169}$$

$$\sum_i\bar{p}_i\bigg[\Big\|\mathscr{U}_{1,t}\big(\hat{\boldsymbol{\rho}}_{S_i}\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)-\frac{1}{\mathscr{N}_i}\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)\mathscr{U}_{1,t}\big(\hat{\boldsymbol{\rho}}_{S_i}\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)\Big\|_1 + \tag{170}$$

$$\Big\|\frac{1}{\mathscr{N}_i}\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)\mathscr{U}_{1,t}\big(\hat{\boldsymbol{\rho}}_{S_i}\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)-\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)\mathscr{U}_{1,t}\big(\hat{\boldsymbol{\rho}}_{S_i}\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)\Big\|_1\bigg] = \tag{171}$$

$$\sum_i\bar{p}_i\bigg[\Big\|\mathscr{U}_{1,t}\big(\hat{\boldsymbol{\rho}}_{S_i}\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)-\frac{1}{\mathscr{N}_i}\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)\mathscr{U}_{1,t}\big(\hat{\boldsymbol{\rho}}_{S_i}\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)\Big\|_1+\Big|\frac{1}{\mathscr{N}_i}-1\Big|\Big\|\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)\mathscr{U}_{1,t}\big(\hat{\boldsymbol{\rho}}_{S_i}\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)\Big\|_1\bigg] = \tag{172}$$

$$\sum_i\bar{p}_i\bigg[\Big\|\mathscr{U}_{1,t}\big(\hat{\boldsymbol{\rho}}_{S_i}\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)-\frac{1}{\mathscr{N}_i}\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)\mathscr{U}_{1,t}\big(\hat{\boldsymbol{\rho}}_{S_i}\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big)\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\big)\Big\|_1+\Big|\frac{1}{\mathscr{N}_i}-1\Big|\mathscr{N}_i\bigg] = \tag{173}$$

$$\sum_i\bar{p}_i\bigg[\bigg\|\Big(\hat{\mathbf{U}}_t\Big(|\psi_{S_i}\rangle\otimes|\psi_{E_0^1}\rangle\Big)\Big)\Big(\hat{\mathbf{U}}_t\Big(|\psi_{S_i}\rangle\otimes|\psi_{E_0^1}\rangle\Big)\Big)^\dagger - \tag{174}$$

$$\bigg(\frac{\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\hat{\mathbf{U}}_t\Big(|\psi_{S_i}\rangle\otimes|\psi_{E_0^1}\rangle\Big)}{\sqrt{\mathscr{N}_i}}|\psi_{S_i}\rangle\otimes|\psi_{E_0^1}\rangle\bigg)\bigg(\frac{\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t^1}\hat{\mathbf{U}}_t\Big(|\psi_{S_i}\rangle\otimes|\psi_{E_0^1}\rangle\Big)}{\sqrt{\mathscr{N}_i}}|\psi_{S_i}\rangle\otimes|\psi_{E_0^1}\rangle\bigg)^\dagger\bigg\|_1+|1-\mathscr{N}_i|\bigg] = \tag{175}$$

$$\sum_i\bar{p}_i\left[\sqrt{1-\left|\frac{\langle\psi_{S_i}|\otimes\langle\psi_E|\hat{\mathbf{U}}_t^\dagger\Big(\mathbb{I}\otimes\hat{\mathbf{P}}_{E_i}\Big)\hat{\mathbf{U}}_t|\psi_{S_i}\rangle\otimes|\psi_E\rangle}{\sqrt{\mathscr{N}_i}}\right|^2}+|1-\mathscr{N}_i|\right] = \tag{176}$$

$$\sum_i \bar{p}_i \left[\sqrt{1 - \left| \frac{\mathcal{N}_i}{\sqrt{\mathcal{N}_i}} \right|^2} + |1 - \mathcal{N}_i| \right] = \tag{177}$$

$$\sum_i \bar{p}_i \left[\sqrt{1 - \mathcal{N}_i} + |1 - \mathcal{N}_i| \right] \leq \tag{178}$$

$$\sum_i \bar{p}_i \left[2\sqrt{1 - \mathcal{N}_i} \right] = 2 \sum_i \bar{p}_i \sqrt{1 - \mathcal{N}_i} \leq \tag{179}$$

$$2\sqrt{\sum_i \bar{p}_i (1 - \mathcal{N}_i)} = \tag{180}$$

$$2\sqrt{\sum_i \bar{p}_i \left(1 - Tr\left\{ \hat{\mathbf{P}}_i^{E_t^1} \Lambda_{t,i}(\hat{\boldsymbol{\rho}}^{E_0^1}) \hat{\mathbf{P}}_i^{E_t^1} \right\} \right)} \tag{181}$$

Going from the (174) and (175) to (176) we have used the following result. Let, $|\psi\rangle$ and $|\phi\rangle$ be two pure state, then

$$\| \psi\rangle\langle\phi| - |\phi\rangle\langle\phi| \|_1 = \sqrt{1 - |\langle\phi|\psi\rangle|^2}. \tag{182}$$

Furthermore, going from (179) to (180) we have used Jensen's inequality for concave functions. In conclusion,

$$\min_{PVM} \left\| \sum_i \left(\hat{\mathbf{P}}_{\Delta_i} \otimes \mathbb{I} \right) \hat{\boldsymbol{\rho}}_t \left(\hat{\mathbf{P}}_{\Delta_i} \otimes \mathbb{I} \right) - \sum_i \left(\hat{\mathbf{P}}_{\Delta_i} \otimes \hat{\mathbf{P}}_i^{E_t^1} \right) \hat{\boldsymbol{\rho}}_t \left(\hat{\mathbf{P}}_{\Delta_i} \otimes \hat{\mathbf{P}}_i^{E_t^1} \right) \right\|_1 \leq \tag{183}$$

$$\min_{PVM} 2\sqrt{\sum_i \bar{p}_i \left(1 - Tr\{ \hat{\mathbf{P}}_i^{E_t^1} \Lambda_{t,i}(\hat{\boldsymbol{\rho}}^{E_0^1}) \hat{\mathbf{P}}_i^{E_t^1} \} \right)} \tag{184}$$

Using Lemma 1 on the latter implies that

$$\min_{PVM} \frac{1}{2} \| \hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{CSBS,t} \|_1 = \tag{185}$$

$$\frac{1}{2} \min_{PVM} \left\| \sum_i \left(\hat{\mathbf{P}}_{\Delta_i} \otimes \mathbb{I} \right) \hat{\boldsymbol{\rho}}_t \left(\hat{\mathbf{P}}_{\Delta_i} \otimes \mathbb{I} \right) - \frac{1}{\mathcal{N}_{tot}} \sum_i \left(\hat{\mathbf{P}}_{\Delta_i} \otimes \hat{\mathbf{P}}_i^{E_t^1} \right) \hat{\boldsymbol{\rho}}_t \left(\hat{\mathbf{P}}_{\Delta_i} \otimes \hat{\mathbf{P}}_i^{E_t^1} \right) \right\|_1 \leq \tag{186}$$

$$2 \min_{PVM} \sqrt{\sum_i \bar{p}_i \left(1 - Tr\{ \hat{\mathbf{P}}_i^{E_t^1} \Lambda_{t,i}(\hat{\boldsymbol{\rho}}^{E_0^1}) \hat{\mathbf{P}}_i^{E_t^1} \} \right)}. \tag{187}$$

Where

$$\mathcal{N}_{tot} = Tr\left\{ \sum_i \left(\hat{\mathbf{P}}_{\Delta_i} \otimes \hat{\mathbf{P}}_i^{E_t^1} \right) \hat{\boldsymbol{\rho}}_t \left(\hat{\mathbf{P}}_{\Delta_i} \otimes \hat{\mathbf{P}}_i^{E_t^1} \right) \right\} = \sum_i \bar{p}_i \mathcal{N}_i. \tag{188}$$

Furthermore, it is important to note that the density operators $\Lambda_{t,i}(\hat{\boldsymbol{\rho}}^{E_0^1})$ are not pure, we may therefore not apply Theorem 1 right away. Before benefiting from Theorem 1 we must separate the purity problem from that of the state discrimination problem. We do this by utilizing the following bound. Defining $\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} := e^{-it\gamma f(x_i) g(\hat{\mathbf{B}})} \hat{\boldsymbol{\rho}}^{E_0^1} e^{it\gamma f(x_i) g(\hat{\mathbf{B}})}$, where $x_i := \int x |\psi_{S_i}(x)|^2 dx$. Now,

$$1 - Tr\{ \hat{\mathbf{P}}_i^{E_t^1} \Lambda_{t,i}(\hat{\boldsymbol{\rho}}^{E_0^1}) \hat{\mathbf{P}}_i^{E_t^1} \} \leq \tag{189}$$

$$\left\| \Lambda_{t,i}(\hat{\boldsymbol{\rho}}^{E_0^1}) - \hat{\mathbf{P}}_i^{E_t^1} \Lambda_{t,i}(\hat{\boldsymbol{\rho}}^{E_0^1}) \hat{\mathbf{P}}_i^{E_t^1} \right\|_1 = \tag{190}$$

$$\left\| \Lambda_{t,i}(\hat{\boldsymbol{\rho}}^{E_0^1}) - \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} + \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} - \hat{\mathbf{P}}_i^{E_t^1} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} \hat{\mathbf{P}}_i^{E_t^1} + \hat{\mathbf{P}}_i^{E_t^1} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} \hat{\mathbf{P}}_i^{E_t^1} - \hat{\mathbf{P}}_i^{E_t^1} \Lambda_{t,i}(\hat{\boldsymbol{\rho}}^{E_0^1}) \hat{\mathbf{P}}_i^{E_t^1} \right\|_1 \leq \tag{191}$$

$$\left\| \Lambda_{t,i}(\hat{\boldsymbol{\rho}}^{E_0^1}) - \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} \right\|_1 + \left\| \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} - \hat{\mathbf{P}}_i^{E_t^1} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} \hat{\mathbf{P}}_i^{E_t^1} \right\|_1 + \left\| \hat{\mathbf{P}}_i^{E_t^1} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} \hat{\mathbf{P}}_i^{E_t^1} - \hat{\mathbf{P}}_i^{E_t^1} \Lambda_{t,i}(\hat{\boldsymbol{\rho}}^{E_0^1}) \hat{\mathbf{P}}_i^{E_t^1} \right\|_1 \leq \tag{192}$$

$$\left\|\Lambda_{t,i}\big(\hat{\boldsymbol{\rho}}^{E_0^1}\big)-\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1}\right\|_1+\left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1}-\hat{\mathbf{P}}_i^{E_t^1}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1}\hat{\mathbf{P}}_i^{E_t^1}\right\|_1+\left\|\Lambda_{t,i}\big(\hat{\boldsymbol{\rho}}^{E_0^1}\big)-\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1}\right\|_1= \tag{193}$$

$$2\left\|\Lambda_{t,i}\big(\hat{\boldsymbol{\rho}}^{E_0^1}\big)-\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1}\right\|_1+\left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1}-\hat{\mathbf{P}}_i^{E_t^1}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1}\hat{\mathbf{P}}_i^{E_t^1}\right\|_1= \tag{194}$$

With this result, we now bound (185).

$$2\min_{PVM}\sqrt{\sum_i\bar{p}_i\Big(1-Tr\big\{\hat{\mathbf{P}}_i^{E_t^1}\Lambda_{t,i}\big(\hat{\boldsymbol{\rho}}^{E_0^1}\big)\hat{\mathbf{P}}_i^{E_t^1}\big\}\Big)}\leq \tag{195}$$

$$2\min_{PVM}\sqrt{\sum_i\bar{p}_i\Big(2\left\|\Lambda_{t,i}\big(\hat{\boldsymbol{\rho}}^{E_0^1}\big)-\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1}\right\|_1+\left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1}-\hat{\mathbf{P}}_i^{E_t^1}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1}\hat{\mathbf{P}}_i^{E_t^1}\right\|_1\Big)}\leq \tag{196}$$

$$2\sqrt{2\sum_i\bar{p}_i\left\|\Lambda_{t,i}\big(\hat{\boldsymbol{\rho}}^{E_0^1}\big)-\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1}\right\|_1}+2\min_{PVM}\sqrt{\sum_i\bar{p}_i\left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1}-\hat{\mathbf{P}}_i^{E_t^1}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1}\hat{\mathbf{P}}_i^{E_t^1}\right\|_1}\Big) \tag{197}$$

We will write the latter result as a theorem.

Theorem 9. *Diagonal terms for continuous variables:*

$$\min_{PVM}\frac{1}{2}\|\hat{\boldsymbol{\rho}}_t-\hat{\boldsymbol{\rho}}_{CSBS,t}\|_1= \tag{198}$$

$$\frac{1}{2}\min_{PVM}\left\|\sum_i\Big(\hat{\mathbf{P}}_{\Delta_i}\otimes\mathbb{I}\Big)\hat{\boldsymbol{\rho}}_t\Big(\hat{\mathbf{P}}_{\Delta_i}\otimes\mathbb{I}\Big)-\frac{1}{\mathcal{N}_{tot}}\sum_i\Big(\hat{\mathbf{P}}_{\Delta_i}\otimes\hat{\mathbf{P}}_{E_i}\Big)\hat{\boldsymbol{\rho}}_t\Big(\hat{\mathbf{P}}_{\Delta_i}\otimes\hat{\mathbf{P}}_{E_i}\Big)\right\|_1\leq \tag{199}$$

$$2\sqrt{2\sum_i\bar{p}_i\left\|\Lambda_{t,i}\big(\hat{\boldsymbol{\rho}}^{E_0^1}\big)-\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1}\right\|_1}+2\min_{PVM}\sqrt{\sum_i\bar{p}_i\left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1}-\hat{\mathbf{P}}_i^{E_t^1}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1}\hat{\mathbf{P}}_i^{E_t^1}\right\|_1} \tag{200}$$

This can be easily extended to the case where we have more than one macro-environment. In such a case Theorem 9 is replaced with the following.

Theorem 10. *Diagonal terms for continuous variables N macro-environment case:*

$$\frac{1}{2}\min_{PVM}\left\|\sum_i\Big(\hat{\mathbf{P}}_{\Delta_i}\otimes\mathbb{I}\Big)\hat{\boldsymbol{\rho}}_t\Big(\hat{\mathbf{P}}_{\Delta_i}\otimes\mathbb{I}\Big)-\frac{1}{\mathcal{N}_{tot}}\sum_i\Big(\hat{\mathbf{P}}_{\Delta_i}\otimes\bigotimes_{k=1}^{N_E}\hat{\mathbf{P}}_i^{E_t^k}\Big)\hat{\boldsymbol{\rho}}_t\Big(\hat{\mathbf{P}}_{\Delta_i}\otimes\bigotimes_{k=1}^{N_E}\hat{\mathbf{P}}_i^{E_t^k}\Big)\right\|_1\leq \tag{201}$$

$$\sqrt{2\sum_i\bar{p}_i\left\|\Lambda_{t,i}\left(\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}^{E_0^k}\right)-\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\right\|_1}+\min_{PVM}\sqrt{\sum_i\bar{p}_i\left\|\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}-\bigotimes_{k=1}^{N_E}\hat{\mathbf{P}}_i^{E_t^k}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\hat{\mathbf{P}}_i^{E_t^k}\right\|_1} \tag{202}$$

Estimating trace distances of the tensor products appearing in the Theorem 10 may be simplified by employing the following lemma.

Lemma 4. *Telescopic inequality:*

$$\|\bigotimes_{k=1}^{N}\hat{\mathbf{A}}^k-\bigotimes_{k=1}^{N}\hat{\mathbf{B}}^k\|_1\leq \tag{203}$$

$$\sum_{j=1}^{N}\left(\prod_{k=1}^{j-1}\|\hat{\mathbf{A}}^k\|_1\right)\times\|\hat{\mathbf{A}}^j-\hat{\mathbf{B}}^j\|_1\times\left(\prod_{k=j+1}^{N}\|\hat{\mathbf{B}}^k\|_1\right) \tag{204}$$

Using Theorem 10 and Lemma 4 we obtain the following useful corollary.

Corollary 2.

$$\frac{1}{2}\min_{PVM}\left\|\sum_i\Big(\hat{\mathbf{P}}_{\Delta_i}\otimes\mathbb{I}\Big)\hat{\boldsymbol{\rho}}_t\Big(\hat{\mathbf{P}}_{\Delta_i}\otimes\mathbb{I}\Big)-\frac{1}{\mathcal{N}_{tot}}\sum_i\Big(\hat{\mathbf{P}}_{\Delta_i}\otimes\bigotimes_{k=1}^{N_E}\hat{\mathbf{P}}_i^{E_t^k}\Big)\hat{\boldsymbol{\rho}}_t\Big(\hat{\mathbf{P}}_{\Delta_i}\otimes\bigotimes_{k=1}^{N_E}\hat{\mathbf{P}}_i^{E_t^k}\Big)\right\|_1\leq \tag{205}$$

$$\sqrt{2\sum_i\bar{p}_i\sum_{k=1}^{N_E}\int|\psi_{S_i}(x)|^2\left\|\hat{\boldsymbol{\rho}}_x^{E_t^k}-\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\right\|_1dx}+\min_{PVM}\sqrt{\sum_i\bar{p}_i\sum_{k=1}^{N_E}\left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}-\hat{\mathbf{P}}_i^{E_t^k}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\hat{\mathbf{P}}_i^{E_t^k}\right\|_1} \tag{206}$$

Proof. First note that

$$\left\|\Lambda_{t,i}\left(\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}^{E_0^k}\right)-\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\right\|_1=\left\|\int|\psi_{S_i}(x)|^2\Big(\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}_x^{E_t^k}\Big)dx-\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\right\|_1= \tag{207}$$

$$\left\|\int|\psi_{S_i}(x)|^2\Big(\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}_x^{E_t^k}-\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\Big)dx\right\|_1\leq \tag{208}$$

$$\int|\psi_{S_i}(x)|^2\left\|\Big(\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}_x^{E_t^k}-\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\Big)\right\|_1dx. \tag{209}$$

Using the latter, the proof follows directly from Lemma 4 and Theorem 9 by noting that

$$\|\hat{\mathbf{P}}_t^{E_t^k}\hat{\boldsymbol{\rho}}_x^{E_t^k}\hat{\mathbf{P}}_t^{E_t^k}\|_1\leq 1 \tag{210}$$

$$\|\hat{\boldsymbol{\rho}}_x^{E_t^k}\|_1\leq 1 \tag{211}$$

for all t, k and x.

□

4 Monitoring CV quantum systems

In the previous section, we have alluded to an analog for SBS states in the continuous variables setting; in this section, we shall formalize SBS theory for continuous variables by developing the notion of quantum monitoring apparatus. We have been studying multipartite systems where one of the subsystems is the system being monitored while the rest are *monitoring devices.* i.e. think of the situation where the system might be a single particle, an atom for instance. This atom may be coupled, through some kind of amplification scheme, to a macroscopic object (the monitoring device/devices) recording information by the position of a "meter" on a dial. The ensemble of the system and monitoring device/devices will be referred to as a *monitoring apparatus*; we formalize this notion below.

Definition 5. *A quantum monitoring apparatus is a septuplet of data*

$$\left(\hat{\boldsymbol{\rho}}_{S_0},\ \bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}^{E_0^k},\ \gamma f(\hat{\mathbf{X}})\otimes\sum_k g_k(\hat{\mathbf{B}}_k),\ \Sigma_t,\ \Lambda_t,\ \mathscr{E}_t,\ T\right)$$

where

- $\hat{\boldsymbol{\rho}}_{s_0}$ *is the initial state of the system at* $t=0$*.*
- $\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}^{E_0^k}$ *is the initial state of the device/devices (*E_0^k *is used as the superscript due to its interpretation as the k-th environment).*
- $\gamma f(\hat{\mathbf{X}})\otimes\sum_k g_k(\hat{\mathbf{B}}_k)$ *is the interaction between the system and the apparatus. We have chosen von Neumann type interaction where all operators* $\hat{\mathbf{X}}$ *and* $\hat{\mathbf{B}}_k$ *are assumed to be either position or momentum operators).* $f(x)$ *and* $g(x)$ *are arbitrary functions.*

- $\Sigma_{k,t}$ *are the resolution limits of the devices defined as* $\Sigma_{k,t} := \left(F_Q\left[\hat{\rho}_x^{E_t^k}, t\gamma g_k(\hat{\mathbf{B}}_k)\right]\right)^{-\frac{1}{2}}$. $\Sigma_t := \max_k \Sigma_{k,t}$ *is the resolution limit of the monitoring apparatus. We will utilize this parameter to bound the size of the partitioning implemented on the state of the system.*

- *In order to define* Λ_t*, the system's resolution, we must first discuss the representation for the state* $\hat{\boldsymbol{\rho}}_{S_0}$ *in the* $\hat{\mathbf{X}}$ *basis. Here* $\hat{X} = \int x|x\rangle\langle x|dx$*, we may express* $\hat{\boldsymbol{\rho}}_{S_0}$ *using Spec*$\{\hat{\mathbf{X}}\}$ *as follows*

$$\hat{\boldsymbol{\rho}}_{S_0} = \int\int \langle x|\hat{\rho}_0|y\rangle|x\rangle\langle y|dxdy.$$

We may partition the integral using the resolution limit as a bound to how small our partitions may be. Let $\{x_{i,t}\}_i$ *be a net in the real line at time* t *with the constraint* $|x_{i,t} - x_{j,t}| > |\Sigma_t|$ *for all* $(x_{i,t}, x_{j,t})$*,* $i \neq j$*, that lie in the support of the kernel of* $\hat{\boldsymbol{\rho}}_{S_0}$*. We create sets* $\Delta_{i,t} := (x_{i,t} - \frac{\sigma_{i,t}}{2}, x_{i,t} + \frac{\sigma_{i,t}}{2})$ *centered at* $x_{i,t}$ *with* $\sigma_{i,t} \geq \Sigma_t$ *for all* t *such that* $|\mathbb{R}\setminus\bigcup_i \Delta_{i,t}| = 0$*. We may now write*

$$\hat{\boldsymbol{\rho}}_{s_0} = \int\int \langle x|\hat{\boldsymbol{\rho}}_{S_0}|y\rangle|x\rangle\langle y|dxdy = \sum_{ij}\int_{\Delta_{i,t}}\int_{\Delta_{j,t}} \langle x|\hat{\boldsymbol{\rho}}_{S_0}|y\rangle|x\rangle\langle y|dxdy := \tag{212}$$

$$\sum_{ij}\sqrt{p_{i,t}p_{j,t}}|\psi_{i,t}\rangle\langle\psi_{j,t}|.$$

where we have defined $|\psi_{i,t}\rangle := \frac{1}{\sqrt{p_{i,t}}}\int_{\Delta_{i,t}}\sqrt{\langle x|\hat{\boldsymbol{\rho}}_{S_0}|x\rangle}|x\rangle dx$ *(utilizing the separability of the kernel* $\langle x|\hat{\boldsymbol{\rho}}_{S_0}|y\rangle$ *and* $p_{i,t} := \int_{\Delta_{i,t}}\langle x|\hat{\boldsymbol{\rho}}_{S_0}|x\rangle dx$ *for normalization.*

We now define the system's resolution with respect to the net $\{x_{i,t}\}$ *at time* t*.*

$$\Lambda_t := \min_{i\neq j; i,j\in\{1,\ldots,n\}}\left|\langle\psi_{i,t}|\hat{\mathbf{X}}|\psi_{i,t}\rangle - \langle\psi_{j,t}|\hat{\mathbf{X}}|\psi_{j,t}\rangle\right|.$$

- $\mathscr{E}_t$ *is a quantum map acting on the system. We will primarily use such maps to model decoherence in the system.*

- T *is the max time of some time domain of monitoring* $[0, T)$*.* T *may be* ∞

Definition 6. *δ-quantum instrument: A δ-quantum instrument will be defined as a* Monitoring Apparatus *satisfying the following additional assumption.*

$$F^2\left(\hat{\boldsymbol{\rho}}_{y_i}^{E_t^k}, \hat{\boldsymbol{\rho}}_{y_j}^{E_t^k}\right) \leq \delta,$$

for all $y_i \in \Delta_{i,t}$ *and* $y_j \in \Delta_{j,t}$ $i \neq j$ *which are also in the support of* $\langle x|\hat{\boldsymbol{\rho}}_{S_0}|x\rangle$*. Where of course* $\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} := e^{-it\gamma f(x_i)g_k(\hat{\mathbf{B}}_k)}\hat{\boldsymbol{\rho}}^{E_0^k}e^{it\gamma f(x_i)g_k(\hat{\mathbf{B}}_k)}$.

A quantum apparatus will be called a δ-quantum instrument for all values of t such that the above conditions are satisfied. We formalize this below and name this time domain the δ-quantum-instrumentalization domain.

$$T_\delta\left(\hat{\boldsymbol{\rho}}_{S_0}, \bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}^{E_0^k}, \gamma f(\hat{\mathbf{X}})\otimes\sum_k g_k(\hat{\mathbf{B}}_k), \Sigma_t, \Lambda_t, \mathscr{E}_t, T\right) := \left\{t < T \middle| F\left(\hat{\boldsymbol{\rho}}_{y_i}^{E_t^k}, \hat{\boldsymbol{\rho}}_{y_j}^{E_t^k}\right) \leq \delta \forall i,j,k \ni i \neq j\right\}.$$

We will simply write T_δ when the elements of the model are understood.

Definition 7. *non-disturbance: Consider the multipartite state*

$$\hat{\rho}_t := \left(e^{-it\gamma f(\hat{\mathbf{X}})\otimes\sum_k g_k(\hat{\mathbf{B}}_k)}\right)\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_0}) \otimes \bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}^{E_0^k}\left(e^{-it\gamma f(\hat{\mathbf{X}})\otimes\sum_k g_k(\hat{\mathbf{B}}_k)}\right).$$

We say that such a state is $\varepsilon-$ non-disturbable at $t > 0$ if there exists a global PVM $\Big\{\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_{j_k}^{E_t^k}\Big\}_{i,j_1,\ldots,j_{N_E}}$ *such that the successful event, i.e. the event corresponding to all of the subsystems coinciding in their post-measurement readings has probability approximately one in the following sense.*

$$\left\| \hat{\boldsymbol{\rho}}_t - \frac{1}{\mathscr{N}_{tot}} \sum_i \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k} \right) \hat{\boldsymbol{\rho}}_t \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k} \right) \right\|_1 < \varepsilon$$

We will name the ε-non-disturbance time domain

$$N_\varepsilon \left(\hat{\boldsymbol{\rho}}_{S_0},\ \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}^{E_0^k},\ \gamma f(\hat{\mathbf{X}}) \otimes \sum_k g_k(\hat{\mathbf{B}}_k),\ \Sigma_t,\ \Lambda_t,\ \mathscr{E}_t,\ T \right) :=$$

$$\left\{ t < T \ \middle|\ \left\| \hat{\boldsymbol{\rho}}_t - \frac{1}{\mathscr{N}_{tot}} \sum_i \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k} \right) \hat{\boldsymbol{\rho}}_t \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k} \right) \right\|_1 < \varepsilon \right\}$$

and write N_ε when the model is understood.

Definition 8. *Good quantum instruments(Haroche): A* good quantum instrument *has been defined by Haroche in [35] as a finite-dimensional-system version of what we call a quantum instrument whose parameters satisfy the following relationship.*

$$\gamma t \Lambda_t \gg \sqrt{Tr\big\{g_k^2(\hat{\mathbf{B}}_k)\hat{\boldsymbol{\rho}}^{E_0^k}\big\} - Tr\big\{g_k(\hat{\mathbf{B}}_k)\hat{\boldsymbol{\rho}}^{E_0^k}\big\}^2},\ \forall\, k$$

What $\gg$ means will depend on a time scale and a desired tolerance. To add precision to Haroche's insightful discussion we will get rid of the $\gg$ symbols and make explicit what we want. Given a chosen time $T_\varepsilon > 0$ associated with the time domain for $\varepsilon-$non-disturbance and a desired tolerance $\delta > 0$, if for all $T \geq t \geq T_\varepsilon$ the parameters of the quantum monitoring apparatus, i.e. γ, Λ_t, $\sqrt{Tr\big\{g_k^2(\hat{\mathbf{B}}_k)\hat{\boldsymbol{\rho}}^{E_0^k}\big\} - Tr\big\{g_k(\hat{\mathbf{B}}_k)\hat{\boldsymbol{\rho}}^{E_0^k}\big\}^2}$, Λ_t, and Σ_t are such that the respective quantum monitoring apparatus is a $\delta-$quantum instrument for a desired portion of the time domain, then the quantum instrument in question is called a good quantum instrument.

Definition 9. *$\varepsilon\delta$- spectrum broadcasting instrument: We will say that a $\delta-$*quantum instrument *is $\varepsilon\delta$-spectrum-broadcasting when it is ε-*non-disturbable. *The $\varepsilon\delta$-spectrum-broadcasting-time domain will be denoted as*

$$ST_{\varepsilon\delta} \left(\hat{\boldsymbol{\rho}}_{S_0},\ \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}^{E_0^k},\ \gamma f(\hat{\mathbf{X}}) \otimes \sum_k g_k(\hat{\mathbf{B}}_k),\ \Sigma_t,\ \Lambda_t,\ \mathscr{E}_t,\ T \right) = N_\varepsilon \cap T_\delta$$

The spectrum broadcasting instruments associated with tolerance parameters $\delta = 0$ and $\varepsilon = 0$ are known as spectrum broadcast structures (SBS) for the case where $Spec\{\hat{\mathbf{X}}\}$ is a discrete set of isolated points and will be called as continuous variables Spectrum Broadcast Structures (CVSBS) for the case where $Spec\{\hat{\mathbf{X}}\}$ has a purely continuous spectrum.

To add intuition to the above definitions, mainly the definition of a quantum monitoring apparatus, a few comments are in order. Although no constraints are imposed on the initial state of the system we would like to comment that the tools developed in this paper are not yet fully sharpened to the extent that they may deal with a general state $\hat{\boldsymbol{\rho}}_{S_0}$. Nevertheless, given that there is no result indicating that a general quantum state may not be a system of some quantum measurement regime type interaction that projects the spectrum of the system onto the environment we are compelled to leave $\hat{\boldsymbol{\rho}}_{S_0}$ as a general state. The same argument goes for the environmental state; although we have provided a bound for quantum state discrimination involving mixtures of mixed states in (1.2), this treatment is only useful for mixtures $\sum_i \hat{\boldsymbol{\rho}}_{i,t}$ where $\hat{\boldsymbol{\rho}}_i = \sum_k \hat{\boldsymbol{\rho}}_{ik,t}$ and all $\hat{\boldsymbol{\rho}}_{ik,t}$ are asymptotically not overlapping for larger t. If more progress were to be made in the quantum state discrimination theory for mixed states one would understand better the emergence of, or lack thereof, SBS structures coming from a more diverse group of initial states. We have focused on Hamiltonians of the type $\gamma f(\hat{\mathbf{X}}) \otimes g(\hat{\mathbf{B}})$ and although our examples only involved the case where $f(x) = x$ and $g(x)$, the case for more general functions $f(x)$ and $g(x)$ is still very much contained within our framework. However, computing the decoherence kernels that come from partial tracing in (11) become quite challenging and one has to apply the theory of stationary phase approximations. We only interested ourselves to the case where the only dynamics considered were the interaction dynamics, this is the quantum measurement limit, where the interaction Hamiltonian has the form $A \otimes B$. It would be interesting to adapt the framework presented in this paper to a more general case, perhaps one in which there is intrinsic dynamics in the system and or environmental degrees of freedom. In the definition of the quantum monitoring apparatus, we define our resolution/partitioning parameter/PVM for the system to be a maximization over all of the local Quantum Fisher Information for each environment. One foresees the Quantum Fisher information to become quite challenging with more general dynamics, how much more difficult would be a great thing to investigate in detail. The resolution parameter depends on a chosen net over the reals. What net to choose in order to partition the system's density matrix will depend on the situation. For example, consider a multi modal distribution $\langle x|\hat{\boldsymbol{\rho}}_{S_0}|x\rangle := |\psi(x)|^2$ with n modes $|\psi(x)|^2$i.e. has n critical numbers $\{x_i\}_{i=1}^n$. If the modes are separated enough and most of the support of each of the modes is smaller than the resolution limit then in such a case we proceed as follows. Define $\Delta_{i,t} := [x_i - \frac{\sigma}{2}, x_i + \frac{\sigma}{2})$, with σ such that $\Delta_{i,t} \bigcap \Delta_{i,t} = \emptyset$ and $\sigma \geq \Sigma \ \forall i \neq j$ in some time domain $t \in [0, T)$ of interest (T could potentially be ∞), where x_i are the critical numbers of $|\psi(x)|^2$ in increasing order and define $\Delta_{n+1,t} := \mathbb{R} \setminus \cup_{i=1}^n \Delta_{i,t}$. We then use this partitioning of the real line to partition the system's state and use this to compute the system's resolution. Heuristically, this parameter computes the smallest gap among the features we are trying to discern. Finally, the quantum maps $\mathscr{E}_t$ we have been studying have been kept quite general, with the exception of the examples.

5 Examples

5.1 System with finite-dimensional Hilbert Space $\mathbb{C}^{d_S}$ and all environments with Hilbert space $L^2(\mathbb{R})$.

Consider the monitoring apparatus of the type $\left(\hat{\boldsymbol{\rho}}_{S_0},\ \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}^{E_0^k},\ \gamma \hat{\mathbf{X}} \otimes \sum_k \hat{\mathbf{B}}_k,\ \Sigma_t,\ \Lambda_t,\ \mathscr{E}_t,\ \infty\right)$

- $\hat{\boldsymbol{\rho}}_{S_0} = \sum_{i,j=1}^{d_S} c_i^* c_j |i\rangle\langle j|$, where $|i\rangle \in \mathbb{C}^{d_S}$ and $\sum_{i=1}^{d_S} |c_i|^2$. We use an orthonormal basis $\{|i\rangle\}_{i=1}^{d_S}$ for $\mathbb{C}^2$ here.
- $\hat{\mathbf{B}} = \sum_{k=1} \hat{\mathbf{B}}_k$, where all B_k moment operators.
- $\hat{\mathbf{X}}$ is the operator $\hat{\mathbf{X}} := \sum_{i=1}^{d_S} x_i |i\rangle\langle i|$, diagonal with respect to the orthonormal basis $\{|i\rangle\}_{i=1}^{d_S}$.
- $f(x) = x$ and $g(x) = x$.
- We will assume the initial environmental state to be a tensor product of macro-environments. Let $\hat{\boldsymbol{\rho}}_{E_0} = \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_{E_0}^k$. We will assume that all of the $\hat{\boldsymbol{\rho}}_{E_0}^k$ are identical. More precisely, we will assume that these states have the following representation with respect to the generalized eigenbasis of the operator conjugate to $\hat{B}_k$ (i.e. the position operator in this case).

$$\hat{\boldsymbol{\rho}}_{E_0}^k = \int\int K_{E_0}(x,y)|x\rangle\langle y| dx dy \tag{213}$$

  where $K_{E_0}(x,y) := \frac{1}{\sigma_{E_0}\sqrt{2\pi}} \exp - \frac{x^2+y^2}{4\sigma_{E_0}^2}$

- The decoherence quantum map $\mathscr{E}_t$ will be defined as follows.

$$\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_0}) = \sum_{i,j=1}^{d_S} c_i^* c_j e^{-\alpha t^2 (x_i - x_j)^2} |i\rangle\langle j| \tag{214}$$

- For the finite-dimensional Hilbert space we do not need a discretization parameter so here there is no Σ_t.
- We assume a good quantum instrument regime with respect to the time regime $t > t_0$, $t_0 := \sqrt{\frac{8\sigma_{E_0}^2 \ln(d_S)}{\gamma^2\Lambda^2}}$ and a quantum-instrumentalization tolerance $\delta > 0$. i.e.

$$F^2\Big(\hat{\boldsymbol{\rho}}_{x_i}^{E_t},\ \hat{\boldsymbol{\rho}}_{x_j}^{E_t}\Big) < \delta,\ \ t_0 < t,\ \ i \neq j. \tag{215}$$

Note that that for pure state $\hat{\boldsymbol{\rho}}_{x_i}^{E_t}$ and $\hat{\boldsymbol{\rho}}_{x_j}^{E_t}$, $F^2\Big(\hat{\boldsymbol{\rho}}_{x_i}^{E_t},\ \hat{\boldsymbol{\rho}}_{x_j}^{E_t}\Big)$ is equivalent to $Tr\big\{\hat{\boldsymbol{\rho}}_{x_i}^{E_t}\hat{\boldsymbol{\rho}}_{x_i}^{E_t}\big\}$ to

Given some $\varepsilon > 0$ our goal is to verify that such a monitoring apparatus enters the $\varepsilon\delta-$spectrum broadcasting instrument regime and to estimate the tail of the $\varepsilon\delta$-spectrum-broadcasting-time-domain. The main hurdle will be to estimate the trace distance defining the non-disturbance criteria Definition 7. In this case

$$\hat{\boldsymbol{\rho}}_t = \Big(e^{-it\gamma\hat{\mathbf{X}}\otimes\sum_{k=1}^{N_E}\hat{\mathbf{B}}_k}\Big)\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}_{S_0}\big)\otimes\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}_{E_0}^k\Big(e^{it\gamma\hat{\mathbf{X}}\otimes\sum_{k=1}^{N_E}\hat{\mathbf{B}}_k}\Big) = \tag{216}$$

$$\sum_{i,j=1}^{d_S} c_i^* c_j e^{-\alpha t^2(x_i-x_j)^2}\Big(e^{-it\gamma\hat{x_i}\otimes\sum_{k=1}^{N_E}\hat{\mathbf{B}}_k}\Big)|i\rangle\langle j|\otimes\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}_{E_0}^k\Big(e^{it\gamma\hat{x_j}\otimes\sum_{k=1}^{N_E}\hat{\mathbf{B}}_k}\Big) \tag{217}$$

$$\sum_{i,j=1}^{d_S} c_i^* c_j e^{-t\alpha t^2(x_i-x_j)^2}|i\rangle\langle j|\otimes\bigotimes_{k=1}^{N_E} e^{-i\gamma\hat{x_i}\hat{\mathbf{B}}_k}\hat{\boldsymbol{\rho}}_{E_0}^k e^{-i\gamma\hat{x_j}\hat{\mathbf{B}}_k} = \tag{218}$$

$$\sum_{i,j=1}^{d_S} c_i^* c_j e^{-\alpha t^2(x_i-x_j)^2}|i\rangle\langle j|\otimes\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}_{i,j}^{E_t^k} \tag{219}$$

We now estimate the corresponding trace distance in Definition 7.

$$\min_{PVM}\left\|\hat{\boldsymbol{\rho}}_t - \frac{1}{\mathscr{N}_{tot}}\sum_{i=1}^{d_S}\left(|i\rangle\langle i|\otimes\bigotimes_{k=1}^{N_E}\hat{\mathbf{P}}_i^{E_t^k}\right)\hat{\boldsymbol{\rho}}_t\left(|i\rangle\langle i|\otimes\bigotimes_{k=1}^{N_E}\hat{\mathbf{P}}_i^{E_t^k}\right)\right\|_1 \leq \tag{220}$$

$$\min_{PVM}\left\|\sum_{i,j}\Big(|i\rangle\langle i|\otimes\mathbb{I}\Big)\hat{\boldsymbol{\rho}}_t\Big(|i\rangle\langle i|\otimes\mathbb{I}\Big) - \frac{1}{\mathscr{N}_{tot}}\sum_{i=1}^{d_S}\left(|i\rangle\langle i|\otimes\bigotimes_{k=1}^{N_E}\hat{\mathbf{P}}_i^{E_t^k}\right)\hat{\boldsymbol{\rho}}_t\left(|i\rangle\langle i|\otimes\bigotimes_{k=1}^{N_E}\hat{\mathbf{P}}_i^{E_t^k}\right)\right\|_1 \leq \tag{221}$$

$$\min_{PVM}\left\|\sum_{i}\Big(|i\rangle\langle i|\otimes\mathbb{I}\Big)\hat{\boldsymbol{\rho}}_t\Big(|i\rangle\langle i|\otimes\mathbb{I}\Big) - \frac{1}{\mathscr{N}_{tot}}\sum_{i=1}^{d_S}\left(|i\rangle\langle i|\otimes\bigotimes_{k=1}^{N_E}\hat{\mathbf{P}}_i^{E_t^k}\right)\hat{\boldsymbol{\rho}}_t\left(|i\rangle\langle i|\otimes\bigotimes_{k=1}^{N_E}\hat{\mathbf{P}}_i^{E_t^k}\right)\right\|_1 + \tag{222}$$

$$\left\|\sum_{i\neq j}\Big(|i\rangle\langle i|\otimes\mathbb{I}\Big)\hat{\boldsymbol{\rho}}_t\Big(|i\rangle\langle i|\otimes\mathbb{I}\Big)\right\|_1 = \tag{223}$$

$$\min_{PVM}\left\|\sum_{i=1}|c_i|^2|i\rangle\langle i|\otimes\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} - \frac{1}{\mathscr{N}_{tot}}\sum_{i=1}|c_i||i\rangle\langle i|\otimes\bigotimes_{k=1}^{N_E}\hat{\mathbf{P}}_i^{E_t^k}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\hat{\mathbf{P}}_i^{E_t^k}\right\|_1 + \tag{224}$$

$$\left\|\sum_{i\neq j} e^{-it\alpha t^2(x_i-x_j)^2}|i\rangle\langle j|\otimes\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}_{i,j}^{E_t^k}\right\|_1 \leq \tag{225}$$

$$\min_{PVM}\sum_{i=1}^{d_S}|c_i|^2\left\||i\rangle\langle i|\otimes\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} - \frac{1}{\mathscr{N}_{tot}}|i\rangle\langle i|\otimes\bigotimes_{k=1}^{N_E}\hat{\mathbf{P}}_i^{E_t^k}\hat{\boldsymbol{\rho}}_{x_{x_i}}^{E_t^k}\hat{\mathbf{P}}_i^{E_t^k}\right\|_1 + \sum_{i\neq j}^{d_S} e^{-\alpha t^2(x_i-x_j)^2}\left\||i\rangle\langle j|\otimes\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}_{i,j}^{E_t^k}\right\|_1 = \tag{226}$$

$$\min_{PVM}\sum_{i=1}^{d_S}|c_i|^2\left\| |i\rangle\langle i|\otimes\left(\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}-\frac{1}{\mathcal{N}_{tot}}\bigotimes_{k=1}^{N_E}\hat{\mathbf{P}}_i^{E_t^k}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\hat{\mathbf{P}}_i^{E_t^k}\right)\right\|_1+\sum_{i\neq j}^{d_S}e^{-\alpha t^2(x_i-x_j)^2}= \tag{227}$$

$$\min_{PVM}\sum_{i=1}^{d_S}|c_i|^2\left\|\bigotimes_{k=1}^{N_E}\hat{\rho}_{x_i}^{E_t^k}-\frac{1}{\mathcal{N}_{tot}}\bigotimes_{k=1}^{N_E}\hat{\mathbf{P}}_i^{E_t^k}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\hat{\mathbf{P}}_i^{E_t^k}\right\|_1+\sum_{i\neq j}^{d_S}e^{-\alpha t^2(x_i-x_j)^2}= \tag{228}$$

$$\min_{PVM}\sum_{i=1}^{d_S}|c_i|^2 2\sum_{k=1}^{N_E}\left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}-\hat{\mathbf{P}}_i^{E_t^k}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\hat{\mathbf{P}}_i^{E_t^k}\right\|_1+\sum_{i\neq j}^{d_S}e^{-\alpha t^2(x_i-x_j)^2}= \tag{229}$$

Going from (228) to (229) we have made use of Lemma 1 and Lemma 4. We have assumed that the environmental terms are all the same, hence we will write $\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}$ as $\hat{\boldsymbol{\rho}}_{x_i}^{E_t}$ for all k in order to emphasize the lack of k dependence. Therefore

$$(229)=2N_E\min_{PVM}\sum_{i=1}^{d_S}|c_i|^2\left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t}-\hat{\mathbf{P}}_i^{E_t}\hat{\boldsymbol{\rho}}_{x_i}^{E_t}\hat{\mathbf{P}}_i^{E_t}\right\|_1+\sum_{i\neq j}^{d_S}e^{-\alpha t^2(x_i-x_j)^2} \tag{230}$$

wherefrom we may implement Theorem 1 in order to obtain a bound on (230). Before proceeding we note we are interested in the large t limit and may therefore pass to the regime where

$$d_S M_{d_S}(t):=d_S\max_{i\neq j;\{1,\ldots,d_S\}}\sqrt{Tr\{\hat{\boldsymbol{\rho}}_{x_i}^{E_t}\hat{\boldsymbol{\rho}}_{x_j}^{E_t}\}}<1. \tag{231}$$

Using Theorem 6 and Corollary 1 it immediately follows that

$$(230)\leq C_t^{d_S,N_E}\sum_{i\neq j}^{d_S}|c_i|^2\sqrt{Tr\{\hat{\boldsymbol{\rho}}_{x_i}^{E_t}\hat{\boldsymbol{\rho}}_{x_j}^{E_t}\}}+\sum_{i\neq j}^{d_S}e^{-\alpha t^2(x_i-x_j)^2} \tag{232}$$

where we have defined $C_t^{d_S,N_E}:=N_E\frac{d_S\left(1+d_S M_{d_S}(t)\right)^{d_S-1}}{\left(1-d_S M_{d_S}(t)\right)^{d_S-1}}$.

For the model at hand, it can be easily shown that

$$\sqrt{Tr\{\hat{\boldsymbol{\rho}}_{x_i}^{E_t}\hat{\boldsymbol{\rho}}_{x_j}^{E_t}\}}=\exp\frac{-\gamma^2t^2(x_i-x_j)^2}{8\sigma_{E_0}^2}. \tag{233}$$

With the latter at hand, we may track the convergence explicitly.

$$(232)=N_E\frac{\left(1+d_S\exp\frac{-\gamma^2t^2\Lambda^2}{8\sigma_{E_0}^2}\right)^{d_S-1}}{\left(1-d_S\exp\frac{-\gamma^2t^2\Lambda^2}{8\sigma_{E_0}^2}\right)^{d_S-1}}d_S^2\exp\frac{-\gamma^2t^2\Lambda^2}{8\sigma_{E_0}^2}+d_S(d_S-1)e^{-\alpha t^2\Lambda^2} \tag{234}$$

Now, let $\varepsilon>0$ and define $t_0:=\sqrt{\frac{8\sigma_{E_0}^2\ln(d_S)}{\gamma^2\Lambda^2}}$. One can easily show that for

$$t>T_\varepsilon:=\max\left\{t_0,\ \sqrt{\frac{8\sigma_{E_0}^2\ln\left(\frac{2d_S C_{t_0}^{d_S,N_E}}{\varepsilon}\right)}{\gamma^2\Lambda^2}},\ \sqrt{\frac{8\sigma_{E_0}^2\ln\left(\frac{2d_S(d_S-1)}{\varepsilon}\right)}{\alpha\Lambda^2}}\ \right\} \tag{235}$$

we have (232) $<\varepsilon$, therefore making this set of values of t a subset of the ε-non-disturbance time domain $N_\varepsilon\left(\hat{\boldsymbol{\rho}}_{S_0},\ \bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}^{E_0^k},\ \gamma\hat{\mathbf{X}}\otimes\sum_{k=1}^{N_E}\hat{\mathbf{B}}_k,\ \Lambda,\ \mathscr{E}_t,\ \infty\right)$. Furthermore, from the good quantum instrument assumptions (215) we know that

$$Tr\{\hat{\boldsymbol{\rho}}_{x_i}^{E_t}\hat{\boldsymbol{\rho}}_{x_j}^{E_t}\}<\delta,\ \ t_0<t. \tag{236}$$

for $i \neq j$. Given that $e^{\frac{-\gamma^2 t^2 \Lambda^2}{8\sigma^2_{E_0}}}$ is monotonically decreasing w.r.t to t it follows that since $T_\varepsilon \geq t_0$

$$Tr\{\hat{\boldsymbol{\rho}}^{E_t}_{x_i}\hat{\boldsymbol{\rho}}^{E_t}_{x_j}\} \leq \exp\frac{-\gamma^2 T_\varepsilon^2 \Lambda^2}{8\sigma^2_{E_0}} < \delta \tag{237}$$

for all $i \neq j$. This means that the time domains $t > T_\varepsilon$ where $T_\varepsilon \geq t_0$ still fall under the good monitoring apparatus regime (215). The time domain $t > T_\varepsilon$ is a subset of the δ-quantum-instrumentalization domain $T_\delta\Big(\hat{\boldsymbol{\rho}}_{S_0},\ \bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}^{E_0^k},\ \gamma\hat{\mathbf{X}}\otimes\sum_{k=1}^{N_E}\hat{\mathbf{B}}_k,\ \Lambda,\ \mathscr{E}_t,\ \infty\Big)$. We therefore conclude that the quantum apparatus $\Big(\hat{\boldsymbol{\rho}}_{S_0},\ \bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}^{E_0^k},\ \gamma\hat{\mathbf{X}}\otimes\sum_{k=1}^{N_E}\hat{\mathbf{B}}_k,\ \Lambda,\ \mathscr{E}_t,\ \infty\Big)$ is $\varepsilon\delta$ spectrum broadcasting for $t > T_\varepsilon$.

Note that for smaller desired values of ε the ε-non-disturbance time domain $t > T_\varepsilon$ shifts to the right due to greater values of T_ε. These greater values of T_ε, in turn, accommodate for smaller values of $\exp\frac{-\gamma^2 T_\varepsilon^2 \Lambda^2}{8\sigma^2_{E_0}}$ leading to more refined quantum-instrumentalization via (237). We, therefore, conclude that in the limit $t \to \infty$ this family of apparatuses in question converges to a spectrum broadcast structure.

5.2 Continuous Variables. The Hilbert spaces of the system and environments respectively will all be $L^2(\mathbb{R})$.

Let us consider the quantum monitoring apparatus $\Big(\hat{\boldsymbol{\rho}}_{S_0},\ \bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}^{E_0^k},\ \gamma\hat{\mathbf{X}}\otimes\hat{\mathbf{B}},\ \Sigma_t,\ \Lambda_t,\ \mathscr{E}_t,\ T\Big)$, where the elements describing the monitoring apparatus are now the following.

- $\hat{\boldsymbol{\rho}}_{s_0}$ is the initial systemic state with corresponding pure state representation ($N_{s_0} = 2$)

$$|\psi_{s_0}\rangle = c_1|\phi_1\rangle + c_2|\phi_2\rangle \ \ (|c_1|^2 + |c_2|^2 = 1) \tag{238}$$

  in function form, this is equivalently

$$\psi_{s_0}(x) = c_1\phi_1(x) + c_2\phi_2(x) \tag{239}$$

  where each function $\phi_i(x) \in C_c^\infty(\mathbb{R})$ with respective supports $\Omega_1 := supp(\phi_1)$ and $\Omega_2 := supp(\phi_2)$. Furthermore, we will assume that the functions $\{\phi_i(x)\}_{i=1}^2$ have disjoint supports.

- We will use the same environmental setting as in the previous example, namely (213).
- $\hat{\mathbf{B}} = \sum_{k=1}\hat{\mathbf{B}}_k$, where all of the operators $\hat{\mathbf{B}}$ are momentum operators
- $\hat{\mathbf{X}}$ is a position operator.
- The decoherence quantum map $\mathscr{E}_t$ will be defined as follows.

$$\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}_{S_0}\big) = \int\int \psi^*_{s_0}(x)\psi_{s_0}(y)e^{-\frac{\gamma^2 t^2 (x-y)^2}{8\sigma^2_{E_0}}}|x\rangle\langle y|dxdy \tag{240}$$

  where we have expressed the operator $\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}_{S_0}\big)$ in the position basis.

- We will use the same partitioning parameter σ for all of the $\Delta_{i,t}$. We will assume that $\sigma \geq \max\{\Sigma_t, |\Omega_1|, |\Omega_2|\}$. We will construct a partition in the following way.$\Delta_i := (x_i - \frac{\sigma}{2},\, x_i + \frac{\sigma}{2})$, x_1 is the midpoint of Ω_1 and x_2 the midpoint of Ω_2 and we assume that $x_1 < x_2$. Note that in this case, the partition to be used will be time-independent, this further implies that Λ will also be time-independent. The rest of the sets Δ_i $i \neq 1, 2$ will be irrelevant since the respective terms of the integral (212) will be zero.

- We finally assume the good quantum instrument regime. i.e. for the parameters of the quantum monitoring apparatus at hand.

Our goal is to verify that such a quantum monitoring apparatus enters the $\varepsilon\delta-$spectrum-broadcasting-instrument regime for an arbitrary $\varepsilon > 0$ and $\delta > 0$. In this case the hurdle of estimating the trace distance defining the non-disturbance criteria (7) is significantly more challenging than it was in the previous section. Here,

$$\min_{PVM} \left\| \hat{\boldsymbol{\rho}}_t - \frac{1}{\mathcal{N}_{tot}} \sum_i \left(\hat{\mathbf{P}}_{\Delta_i} \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k} \right) \hat{\boldsymbol{\rho}}_t \left(\hat{\mathbf{P}}_{\Delta_i} \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k} \right) \right\|_1 \leq \tag{241}$$

$$\min_{PVM} \left\| \sum_{i,j} \left(\hat{\mathbf{P}}_{\Delta_i} \otimes \mathbb{I} \right) \hat{\boldsymbol{\rho}}_t \left(\hat{\mathbf{P}}_{\Delta_j} \otimes \mathbb{I} \right) - \frac{1}{\mathcal{N}_{tot}} \sum_i \left(\hat{\mathbf{P}}_{\Delta_i} \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k} \right) \hat{\boldsymbol{\rho}}_t \left(\hat{\mathbf{P}}_{\Delta_i} \otimes \bigotimes_{k=1}^{N} \hat{\mathbf{P}}_i^{E_t^k} \right) \right\|_1 \leq \tag{242}$$

$$\min_{PVM} \left\| \sum_{i} \left(\hat{\mathbf{P}}_{\Delta_i} \otimes \mathbb{I} \right) \hat{\boldsymbol{\rho}}_t \left(\hat{\mathbf{P}}_{\Delta_i} \otimes \mathbb{I} \right) - \frac{1}{\mathcal{N}_{tot}} \sum_i \left(\hat{\mathbf{P}}_{\Delta_i} \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k} \right) \hat{\boldsymbol{\rho}}_t \left(\hat{\mathbf{P}}_{\Delta_i} \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k} \right) \right\|_1 + \tag{243}$$

$$\left\| \sum_i \sum_{j;j\neq i} \left(\hat{\mathbf{P}}_{\Delta_i} \otimes \mathbb{I} \right) \hat{\boldsymbol{\rho}}_t \left(\hat{\mathbf{P}}_{\Delta_j} \otimes \mathbb{I} \right) \right\|_1 \leq \tag{244}$$

$$\sqrt{2N_E\sum_i \bar{p}_i\int |\psi_{s_i}(x)|^2\left\|\hat{\boldsymbol{\rho}}_x^{E_t}-\hat{\boldsymbol{\rho}}_{x_i}^{E_t}\right\|_1 dx}+\min_{PVMP_{E_i}}\sqrt{N_E\sum_i \bar{p}_i\left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t}-\hat{\mathbf{P}}_i^{E_t}\hat{\boldsymbol{\rho}}_{x_i}^{E_t}\hat{\mathbf{P}}_i^{E_t}\right\|_1}+ \quad (245)$$

$$\sum_i\sum_{j;j\neq i}\|\hat{\mathbf{P}}_{\Delta_i}\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_0})\hat{\mathbf{P}}_{\Delta_j}\|_1 \quad (246)$$

where we have used Corollary 2 and equations (113) through (118) in the final term of the above sequence of inequalities. We also made use of the fact that all of the environmental degrees of freedom are the same and all $\hat{\mathbf{B}}_k$ are the same. To accentuate the latter we wrote $\hat{\boldsymbol{\rho}}_x^{E_t}=\hat{\boldsymbol{\rho}}_x^{E_t^k}$ and $\hat{\mathbf{P}}_i^{E_t}=\hat{\mathbf{P}}_i^{E_t^k}$ for all k to accentuate the k independence. We remind the reader that the term $\bar{p}_i:=\int_{\Delta_i}|\psi_{S_0}(x)|^2dx$.

We will now treat each one of the terms in (245) and (246) independently. We will begin with the sum pertaining to the non-diagonal terms, i.e. $\sum_i\sum_{j;j\neq i}\|\hat{\mathbf{P}}_{\Delta_i}\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_0})\hat{\mathbf{P}}_{\Delta_j}\|_1$. Sections 3.1 and 3.2 tell us how to bound such a trace distance. In this case, the corresponding off-diagonal terms will be bounded by using the Kupsch-like bounds derived in section (3.1).

$$\sum_i\sum_{j;j\neq i}\left\|\hat{\mathbf{P}}_{\Delta_i}\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_0})\hat{\mathbf{P}}_{\Delta_j}\right\|_1\leq\sum_i\sum_{j;j\neq i}\sup_{(x,y)\in\Delta_i\times\Delta_j}\left(2\left|\exp\frac{-\gamma^2t^2(x-y)^2}{8\sigma_{E_0}^2}\right|+\sigma\left|\partial_y\exp\frac{-\gamma^2t^2(x-y)^2}{8\sigma_{E_0}^2}\right|\right)= \quad (247)$$

$$\sum_i\sum_{j;j\neq i}\sup_{(x,y)\in\Delta_i\times\Delta_j}\left(2\exp\frac{-\gamma^2t^2(x-y)^2}{8\sigma_{E_0}^2}+\frac{\gamma^2t^2\sigma}{8\sigma_{E_0}^2}|x-y|\exp\frac{-\gamma^2t^2(x-y)^2}{8\sigma_{E_0}^2}\right). \quad (248)$$

For this case $N_{s_0}=2$ the supports of the ϕ_i are not overlapping, $|\Delta_i|\geq|\Omega_i|,\ i=1,2$. Therefore,

$$(248)=2\sup_{(x,y)\in\Delta_1\times\Delta_2}\left(2\exp\frac{-\gamma^2t^2(x-y)^2}{8\sigma_{E_0}^2}+\frac{\gamma^2t^2\sigma}{4\sigma_{E_0}^2}|x-y|\exp\frac{-\gamma^2t^2(x-y)^2}{8\sigma_{E_0}^2}\right)\leq \quad (249)$$

$$4\exp\frac{-\gamma^2t^2\bar{\Lambda}_1^2}{4\sigma_{E_0}^2}\left(1+\frac{\gamma^2t^2\bar{\Lambda}_2\sigma}{8\sigma_{E_0}^2}\right) \quad (250)$$

where (249) is so because the terms we are taking the supremum over are symmetric with respect to exchanges of x with y and vice-versa. In (250), $\bar{\Lambda}_1:=\min\{\Delta_2\}-\max\{\Delta_1\}$ while $\bar{\Lambda}_2:=\max\{\Delta_2\}-\min\{\Delta_1\}$.

Let us now concentrate on the diagonal terms. We will begin with the optimization term.

$$\min_{PVM}\sqrt{N_E\sum_{i=1}^{N_{S_0}}\bar{p}_i\left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t}-\hat{\mathbf{P}}_i^{E_t}\hat{\boldsymbol{\rho}}_{x_i}^{E_t}\hat{\mathbf{P}}_i^{E_t}\right\|_1}= \quad (251)$$

$$\min_{PVM}\sqrt{N_E\sum_{i=1}^{2}\bar{p}_i\left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t}-\hat{\mathbf{P}}_i^{E_t}\hat{\boldsymbol{\rho}}_{x_i}^{E_t}\hat{\mathbf{P}}_i^{E_t}\right\|_1}\leq \quad (252)$$

$$\sqrt{2N_E Tr\{\hat{\boldsymbol{\rho}}_{x_1}^{E_t}\hat{\boldsymbol{\rho}}_{x_2}^{E_t}\}}= \quad (253)$$

$$\sqrt{2N_E\exp\frac{-\gamma^2t^2(x_1-x_2)^2}{8\sigma_{E_0}^2}}=\sqrt{2N_E}\exp\frac{-\gamma^2t^2(x_1-x_2)^2}{16\sigma_{E_0}^2}\leq\sqrt{2N_E}\exp\frac{-\gamma^2t^2\bar{\Lambda}_1^2}{16\sigma_{E_0}^2} \quad (254)$$

Here we have used Theorem 1, which for the $d_S=2$ case is simple to compute. For d_S much larger we would have resorted to using Corollary 1. Finally, we bound the remaining diagonal term. The purity error of the states $\int|\phi_i(x)|^2\hat{\boldsymbol{\rho}}_x^{E_t}$.

$$\sqrt{2N_E\sum_{i=1}^{2}\bar{p}_i\int|\phi_i(x)|^2\left\|\hat{\boldsymbol{\rho}}_x^{E_t}-\hat{\rho}_{x_i}^{E_t}\right\|_1dx}\leq \quad (255)$$

$$\sqrt{2N_E\sum_{i=1}^{2}\bar{p}_i\int|\phi_i(x)|^2\sqrt{1-Tr\{\hat{\boldsymbol{\rho}}_x^{E_t}\hat{\boldsymbol{\rho}}_{x_i}^{E_t}\}}dx} = \tag{256}$$

$$\sqrt{2N_E\sum_{i=1}^{2}\bar{p}_i\int|\phi_i(x)|^2\sqrt{1-\exp\frac{-\gamma^2t^2(x-x_i)^2}{8\sigma_{E_0}^2}}dx} \leq \sqrt{2N_E\sqrt{1-\exp\frac{-\gamma^2t^2\lambda^2}{8\sigma_{E_0}^2}}} \tag{257}$$

with λ defined as $\max_{i=1,2}|\Omega_i|$.

With all of the terms now estimated we recapitulate with the bound below.

$$\min_{PVM}\left\|\hat{\boldsymbol{\rho}}_t-\frac{1}{\mathcal{N}_{tot}}\sum_i\left(\hat{\mathbf{P}}_{\Delta_i}\otimes\bigotimes_{k=1}^{N_E}\hat{\mathbf{P}}_i^{E_t^K}\right)\hat{\boldsymbol{\rho}}_t\left(\hat{\mathbf{P}}_{\Delta_i}\otimes\bigotimes_{k=1}^{N_E}\hat{\mathbf{P}}_i^{E_t^k}\right)\right\|_1 \leq \tag{258}$$

$$4\exp\frac{-\gamma^2t^2\bar{\Lambda}_1^2}{8\sigma_{E_0}^2}\left(1+\frac{\gamma^2t^2\sigma\bar{\Lambda}_2}{4\sigma_{E_0}^2}\right)+\sqrt{2N_E}\exp\frac{-\gamma^2t^2\bar{\Lambda}_1^2}{16\sigma_{E_0}^2}+\sqrt{2N_E\sqrt{1-\exp\frac{-\gamma^2t^2\lambda^2}{8\sigma_{E_0}^2}}} \tag{259}$$

The parameters $\bar{\Lambda}_1$ and $\bar{\Lambda}_2$ have the following bounds.

$$\bar{\Lambda}_1 \geq \Lambda-2\lambda,\ \ \textit{while}\ \ \bar{\Lambda}_2 \leq \Lambda+\lambda \tag{260}$$

Proof. Let $\mu_1 := \int_{\Omega_1}x|\phi(x)|^2dx$, and $\mu_2 := \int_{\Omega_2}x|\phi(x)|^2dx$. Then,

$$\bar{\Lambda}_1 = \min\left\{\Delta_2\right\}-\max\left\{\Delta_1\right\} \geq \tag{261}$$

$$\min\left\{\Delta_2\right\}-\max\left\{\Delta_1\right\}-\left(\left(\max\left\{\Delta_1\right\}-\mu_1\right)+\left(\mu_2-\min\left\{\Delta_2\right\}\right)\right) \geq \tag{262}$$

$$\min\left\{\Delta_2\right\}-\max\left\{\Delta_1\right\}+\left(\lambda+\lambda\right) \tag{263}$$

This gives us $\bar{\Lambda}_1 \geq \Lambda-2\lambda$. To get $\bar{\Lambda}_2 \leq \Lambda+\lambda$ one implements the same simple techniques. □

Hence,

$$(259) \leq 4\exp\frac{-\gamma^2t^2\Lambda^2(1-2\frac{\lambda}{\Lambda})^2}{8\sigma_{E_0}^2}\left(1+\frac{\gamma^2t^2\sigma\Lambda(1+2\frac{\lambda}{\Lambda})}{4\sigma_{E_0}^2}\right)+ \tag{264}$$

$$\sqrt{2N_E}\exp\frac{-\gamma^2t^2\Lambda^2(1-2\frac{\lambda}{\Lambda})^2}{16\sigma_{E_0}^2}+\sqrt{2N_E\sqrt{1-\exp\frac{-\gamma^2t^2\lambda^2}{8\sigma_{E_0}^2}}} \tag{265}$$

To further analyze the bound (264),(265) with more ease we introduce the variable $\tau := \frac{\gamma t\Lambda}{\sigma_{E_0}}$. The latter is now written and bounded as follows.

$$(259) \leq 4\exp\frac{-\tau^2(1-2\frac{\lambda}{\Lambda})^2}{8}\left(1+\frac{\tau^2\sigma(1+2\frac{\lambda}{\Lambda})}{4\Lambda}\right)+ \tag{266}$$

$$\sqrt{2N_E}\exp\frac{-\tau^2(1-2\frac{\lambda}{\Lambda})^2}{16}+\sqrt{2N_E\sqrt{1-\exp\frac{-\tau^2\lambda^2}{8\Lambda^2}}} \leq \tag{267}$$

$$4e^{\frac{-\tau^2(1-\frac{2\sigma}{\Lambda})^2}{8}}\left(1+\frac{\tau^2\sigma(1+\frac{2\sigma}{\Lambda})}{4\Lambda}\right)+ \tag{268}$$

$$\sqrt{2N_E}e^{\frac{-\tau^2(1-\frac{2\sigma}{\Lambda})^2}{16}}+\sqrt{2N_E\sqrt{1-e^{\frac{-\tau^2\sigma^2}{8\Lambda^2}}}} \leq \tag{269}$$

$$e^{\frac{-\tau^2(1-\frac{2\sigma}{\Lambda})^2}{16}}\left(4+\sqrt{2N_E}+\frac{\tau^2\sigma}{\Lambda}\left(1+\frac{2\sigma}{\Lambda}\right)\right)+\sqrt{2N_E\sqrt{1-e^{\frac{-\tau^2\sigma^2}{8\Lambda^2}}}} \leq \tag{270}$$

$$e^{\frac{-\tau^2(1-\frac{2\sigma}{\Lambda})^2}{16}}\Big(6\sqrt{N_E}+\frac{3\tau^2\sigma}{\Lambda}\Big)+\sqrt{2N_E\sqrt{1-e^{\frac{-\tau^2\sigma^2}{8\Lambda^2}}}}= \tag{271}$$

$$6\sqrt{N_E}e^{\frac{-\tau^2(1-\frac{2\sigma}{\Lambda})^2}{16}}\Big(1+\frac{\tau^2\sigma}{2\Lambda\sqrt{N_E}}\Big)+\sqrt{2N_E\sqrt{1-e^{\frac{-\tau^2\sigma^2}{8\Lambda^2}}}}\leq \tag{272}$$

$$6\sqrt{N_E}e^{\frac{-\tau^2(1-\frac{2\sigma}{\Lambda})^2}{16}}e^{\frac{\tau^2\sigma}{2\Lambda\sqrt{N_E}}}+\sqrt{2N_E\sqrt{1-e^{\frac{-\tau^2\sigma^2}{8\Lambda^2}}}}= \tag{273}$$

$$6\sqrt{N_E}e^{-\tau^2\Big((\frac{1}{4}-\frac{\sigma}{2\Lambda})^2-\frac{\sigma}{2\Lambda\sqrt{N_E}}\Big)}+\sqrt{2N_E\sqrt{1-e^{\frac{-\tau^2\sigma^2}{8\Lambda^2}}}}. \tag{274}$$

Now, take $1>\delta>0$ and $1>\epsilon>0$. For our quantum monitoring apparatus to be a δ- quantum instrument (6) we require that

$$Tr\big\{\hat{\boldsymbol{\rho}}_{x_1}^{E_t}\hat{\boldsymbol{\rho}}_{x_2}^{E_t}\big\}\leq\delta,\ \ \forall\,(x_1,x_2)\in\Omega_1\times\Omega_2. \tag{275}$$

It can easily be shown that this holds for

$$\tau\geq\sqrt{\frac{-8\ln\delta}{(1-\frac{2\sigma}{\Lambda})^2}}. \tag{276}$$

Furthermore, if we would like the quantum monitoring apparatus to be ε-non-disturbable as well, then the following must also be satisfied.

$$6\sqrt{N_E}e^{-\tau^2\Big((\frac{1}{4}-\frac{\sigma}{2\Lambda})^2-\frac{\sigma}{2\sqrt{N_E}\Lambda}\Big)}+\sqrt{2N_E\sqrt{1-e^{\frac{-\tau^2\sigma^2}{8\Lambda^2}}}}\leq\varepsilon \tag{277}$$

The latter implies the following two bounds on τ.

$$\tau\leq\frac{\Lambda}{\sigma}\sqrt{-8\ln\Big(1-\Big(\frac{\varepsilon^2}{8N_E}\Big)^2\Big)} \tag{278}$$

$$\tau\geq\sqrt{\frac{-\ln\frac{\varepsilon}{12\sqrt{N_E}}}{\Big((\frac{1}{4}-\frac{\sigma}{\Lambda})^2-\frac{\sigma}{2\Lambda\sqrt{N_E}}\Big)}} \tag{279}$$

Changing back to $\tau=\frac{\gamma t\Lambda}{\sigma_{E_0}}$ the inequalities (276)(278)(279) have the following form.

$$t\geq\frac{\sigma_{E_0}}{\gamma\Lambda}\sqrt{\frac{-8\ln\delta}{(1-\frac{2\sigma}{\Lambda})^2}}. \tag{280}$$

$$t\geq\frac{\sigma_{E_0}}{\gamma\Lambda}\sqrt{\frac{-\ln\frac{\varepsilon}{12\sqrt{N_E}}}{\Big((\frac{1}{4}-\frac{\sigma}{\Lambda})^2-\frac{\sigma}{2\Lambda\sqrt{N_E}}\Big)}} \tag{281}$$

$$t\leq\Big(\frac{\Lambda}{\sigma}\Big)\frac{\sigma_{E_0}}{\gamma\Lambda}\sqrt{-8\ln\Big(1-\Big(\frac{\varepsilon^2}{8N_E}\Big)^2\Big)} \tag{282}$$

In general, such a system of inequalities will not be satisfied. For the above system of inequalities to hold, the following inequality must be satisfied.

$$\max\left\{\frac{\sqrt{\frac{-8\ln\delta}{(1-\frac{2\sigma}{\Lambda})^2}}}{\sqrt{-8\ln\Big(1-\Big(\frac{\varepsilon^2}{8N_E}\Big)^2\Big)}},\ \frac{\sqrt{\frac{-\ln\frac{\varepsilon}{12\sqrt{N_E}}}{\Big((\frac{1}{4}-\frac{\sigma}{\Lambda})^2-\frac{\sigma}{2\Lambda\sqrt{N_E}}\Big)}}}{\sqrt{-8\ln\Big(1-\Big(\frac{\varepsilon^2}{8N_E}\Big)^2\Big)}}\right\}\leq\frac{\Lambda}{\sigma} \tag{283}$$

If (283) is satisfied for some prescribed Λ and σ, and if $\frac{\Lambda}{\sigma}$ is large enough so that

$$\left(\frac{\Lambda}{\sigma}\right)\frac{\sigma_{E_0}}{\gamma\Lambda}\sqrt{-8\ln\left(1-\left(\frac{\varepsilon^2}{8N_E}\right)^2\right)} \geq T \tag{284}$$

with $\frac{\sigma_{E_0}}{\gamma\Lambda}$ small, then our quantum monitoring apparatus is a $\varepsilon\delta$-spectrum broadcasting-instrument. Furthermore, the associated $\varepsilon\delta$-spectrum-broadcasting-time $ST_{\varepsilon\delta}$ is then given by the inequalities (280)(281)(282).

6 Future work

$$\partial_t \rho_S(t) = \frac{-i}{2}\Delta\omega_a[\sigma_z, \rho_S(t)] + \gamma D[\sigma_-]\rho_S(t). \tag{285}$$

this concludes our work. Equation (285) is the master equaiton of the two-level atom spontaneous emission model.

References

[1] Reed. Michael, Simon. Barry *Funtional Analysis volume one* . Academic Press 1980.

[2] Shlomo Sternberg *A mathematical companion to Quantum Mechanics* . Dover Publications, inc 2019.

[3] Barry Simon *The Theory of Scrhödinger Operators: Waht's It All About?*

[4] Townsend John S. *A modern Approach to QUANTUM MECHANICS 2nd edition* . University Science Books, 2012 .

[5] Hollowood Timothy J.*Coppenhagen Quantum Mechanics.* Arxiv article, 2015

[6] A. Galindo, P.Pascual *Quantum Mechanics 1*, (Pringer-Verlag Berlin Heidelberg 1990).

[7] M.A. Nielsen , I.L. Chuang *Quantum Computation and Quantum Information*, (Cambridge University Press, Cambridge, 10th edition, 2011).

[8] J.J. Sakurai„ Jim J. Napolitano. *Modern Quantum Mechanics*, (Pearson. Editor, (2nd edition). (2014)).

[9] M. Schlosshauer *Decoherence and the Quantum-To-Classical Transition*, (Springer-Verlag, Berlin Heidelberg, 2007).

[10] E.B.Manoukian *Quantum Theory A Wide Spectrum*, (Springer 2006).

[11] K. Fuji *Introduction to the Rotating Wave Approximation, (RWA) :Two Coherent Oscillations*, (International College of Arts and Sciences, Yokohama City University, Yokohama, Japan 2014.)

[12] H. M. Wiseman, G. J. Milburn *Quantum Measurement and Control*, (Cambridge University Press, Cambridge, 2009).

[13] J.R.Retherford *Hilbert Space: Compact Operators and the Trace Theorem*, (Cambridge University Press 1993).

[14] R. ALicki, K. Lendi *Quantum Dynamical Semigroups and Applications, 2nd Edition, Vol.717 of Lect. Notes Phys.*, (Springer, Berlin/Heidelberg, 2007.)

[15] M. Schlosshauer *Quantum Decoherence.*, (Department of Physics University of Portland,Portland USA, 2019.)

[16] E. Joos, H.D. Zeh *The emergence of classical properties through interaction with the environment.*, (Z.Phisik B- Condensed Matter 59, 223-243, 1985.)

[17] Moodley Mervlyn, Petruccione Francesco *Stochastic Wave-Function Unravelling of the Generalized Lindblad Master Equation.*, (Arxiv 2009)

[18] John Hunter, Bruno Nachtergaele *Applied Analysis*, (Course notes 2005)

[19] Castin Yvan, Dalibard Jean. *A Wave Function approach to dissipative processes.*, (Arxiv 2008)

[20] R.Horodecki, J.K.Korbicz, and P.Horodecki *Quantum origins of objectivity.*, (Phys. Rev. A91, 032122- Published 30 March 2015.)

[21] Kornyik, M. and Vukics, A. (2019) *The Monte Carlo wave-function method: A robust adaptive algorithm and a study in convergence.* (COMPUTER PHYSICS COMMUNICATIONS, 238. pp. 88-101. ISSN 0010-4655)

[22] Peter D.Lax *Functional Analysis.* (John Wiley and Sons Inc., New York, 2002).

[23] M. Schlosshauer *Quantum Decoherence* (Department of Physics University of Portland, Portland USA 2019).

[24] A Narasimhan, Chopra Deepak, Kafatos Menas C. (2019) *The Nature of the Heisenberg-von Neumann Cut* (Act Nerv Super 61, 12–17 (2019))

[25] A Narasimhan, Chopra Deepak, Kafatos Menas C. (2019) *IBM Quantum Computing Website.* (https://www.ibm.com/quantum-computing)

[26] Chruscinski Dariusz , Pascazio, Saverio *"A brief History of the GKSL Equation", Based on a talk given by D.C. at the 48th Symposium on Mathematical Physics "Gorini-Kossakowski-Lindblad-Sudarshan Master Equation*

[27] R. Horodecki, J. K. Korbicz, and P. Horodecki *Quantum origins of objectivity.*Phys. Rev. A 91, 032122 – Published 30 March 2015

[28] J.Tuziemski and J.K.Korbicz *Dynamical objectivity in quantum Brownian motion.* Europhysics Letters, Volume 112, Number 4

[29] P. Mironowicz, J.K. Korbicz, and P. Horodecki *Monitoring of the Process of System Information Broadcasting in Time.* Phys. Rev. Lett. 118, 150501 – Published 10 April 2017

[30] Wojciech Hubert Zurek *Quantum Darwinism.* Nature Physics, vol. 5, pp. 181-188 (2009)

[31] C.Jess Riedel, Wojciech H. Zurek *Quantum Darwinism in an Everyday Environment: Huge Redundancy in Scattered Photons* . Phys. Rev. Lett. 105, 020404 (2010)

[32] Thao P. Le and Alexandra Olaya-Castro *Strong Quantum Darwinism and Strong Independence are Equivalent to Spectrum Broadcast Structure* . Phys. Rev. Lett. 122, 010403 – Published 8 January 2019

[33] K. Yosida *Functional Analisis.* Springer-Verlag, 1995.

[34] Townsend John S. *A modern Approach to QUANTUM MECHANICS 2nd edition* . University Science Books, 2012 .

[35] Serge-Haroche and Jean-Michel Raimond. *Exploring the Quantum; Atoms, Cavities, and photons*, (Oxford Gradate Texts, New York, 1st edition, 2006).

[36] Hollowood Timothy J.*Coppenhagen Quantum Mechanics.* Arxiv article, 2015

[37] M.A. Nielsen , I.L. Chuang *Quantum Computation and Quantum Information*, (Cambridge University Press, Cambridge, 10th edition, 2011).

[38] Joachim, Kupsch *The role of infrared divergence for decoherence*, (J. Math. Phys. 41, 5945 (2000))

[39] M. Schlosshauer *Decoherence and the Quantum-To-Classical Transition*, (Springer-Verlag, Berlin Heidelberg, 2007).

[40] E. Joos, H.D. Zeh *The emergence of classical properties through interaction with the environment.*, (Z.Phisik B- Condensed Matter 59, 223-243, 1985.)

[41] B. Simon *Trace Ideals and Their Applications*, (American Mathematical Society and Monographs Volume 120 (1979)).

[42] K. Fuji *Introduction to the Rotating Wave Approximation, (RWA) :Two Coherent Oscillations*, (International College of Arts and Sciences, Yokohama City University, Yokohama, Japan 2014.)

[43] H. M. Wiseman, G. J. Milburn *Quantum Measurement and Control*, (Cambridge University Press, Cambridge, 2009).

[44] R. ALicki, K. Lendi *Quantum Dynamical Semigroups and Applications, 2nd Edition, Vol.717 of Lect. Notes Phys.*, (Springer, Berlin/Heidelberg, 2007.)

[45] Gorino, V., Kossakowski, A. Sudarshan, E.C.G. (1976) *Completely Positive Dynamical Semigroups of N-Level Systems*, (Journal of Mathematical Physics, 17,821-825)

[46] I. Siemon, A.S. Holevo, R.F.Werner *Unbounded generators of dynamical semigroups*, (Open Systems and Information Dynamics Vol.24, No.4 (2017) 1740015, World Scientific Publishing Company)

[47] M. Schlosshauer *Quantum Decoherence.*, (Department of Physics University of Portland,Portland USA, 2019.)

[48] E. Joos, H.D. Zeh *The emergence of classical properties through interaction with the environment.*, (Z.Phisik B- Condensed Matter 59, 223-243, 1985.)

[49] Moodley Mervlyn, Petruccione Francesco *Stochastic Wave-Function Unravelling of the Generalized Lindblad Master Equation.*, (Arxiv 2009)

[50] Stanford S. Bonan and Dean S.Clark. *Estimates of the Hermite and the Freud Polynomials.*, (Journal of Approximation Theory Volume 63, issue 2, November 1990, pages 210-224)

[51] W.H.Zurek *Pointer basis of quantum apparatus: Into what mixture does the wave packet collapse?* , (Phys. Rev. D 24 (1981) 1516-1526)

[52] J.K.Korbicz. *Road to objectivity: Quantum Darwinim, SPectrum Broadcast Structures, and Strong quantum Darwinism — a review.*, (Arxiv 2008)

[53] L.Trefethen, D.Bau . *Numerical Linear Algebra*, (SIAM, (1997))

[54] Castin Yvan, Dalibard Jean. *A Wave Function approach to dissipative processes.*, (Arxiv 2008)

[55] Kornyik, M. and Vukics, A. (2019) *The Monte Carlo wave-function method: A robust adaptive algorithm and a study in convergence.* (COMPUTER PHYSICS COMMUNICATIONS, 238. pp. 88-101. ISSN 0010-4655)

[56] Caves, Carlton M. ; Milburn, G. J.(1987) Quantum-mechanical model for continuous position measurements (Physical Review A (General Physics), Volume 36, Issue 12, December 15, 1987, pp.5543-5555)

[57] Michele Dall'Arno, Giacomo Mauro D'Ariano, and Massimiliano F. Sacchi. (2011) *Informational power of quantum measurements* (Phys. Rev. A 83, 06230 (2011))

[58] H. Barnum, E. Knill (2000) *Reversing quantum dynamics with near-optimal quantum and classical fidelity AIP Journal of Mathematical Physics (2002)*

[59] A Narasimhan, Chopra Deepak, Kafatos Menas C. (2019) *The Nature of the Heisenberg-von Neumann Cut* (Act Nerv Super 61, 12–17 (2019))

[60] Joachim Kupsch C. (2000) *The role of infrared divergence for decoherence* (J. Math. Phys. 41, 5945 (2000))

[61] A Narasimhan, Chopra Deepak, Kafatos Menas C. (2019) *IBM Quantum Computing Website.* (https://www.ibm.com/quantum-computing)

[62] Chruscinski Dariusz , Pascazio, Saverio *"A brief History of the GKSL Equation"*, *Based on a talk given by D.C. at the 48th Symposium on Mathematical Physics "Gorini-Kossakowski-Lindblad-Sudarshan Master Equation*

[63] Michael Reed, Barry Simon , *Functional Analysis* (Academic Press Limited 1980).

[64] Hongwei Sun , *Mercer theorem for RKHS on noncompact sets* (Journal of Complexity Volume 21, Issue 3, June 2005, Pages 337-349).

[65] Yogesh J.Bagul, Christophe Chesneau *Sigmoid functions for the smooth approximation to the absolute value function* (Moroccan J. of Pure and Appl. Anal.(MJPAA), Volume Issue: Volume 7 (2021) - Issue 1 (January 2021)).

[66] Montanaro, Ashley *Pretty simple bounds on quantum state discrimination* (August 2019 Arxiv https://arxiv.org/abs/1908.08312).

[67] A. Nielsen Michael, L. Chuang Isaac *Quantum Computation and Quantum Information* (Cambridge University Press 2010).