Asymptotic Quantum State Discrimination for both, countable, and uncountable unitarily related mixtures.

Alberto Acevedo, Janek Wehr

The University of Arizona, Tucson, USA

Abstract: Given a mixed state, finding a POVM that optimally discriminates between the elements of the mixture is a prominent problem in quantum communications theory. In this paper, we will address mixtures of density operators which are unitarily similar in the asymptotic regime, with respect to a dynamics-parameter parameter and their respective quantum state discrimination (QSD) problems. In the past little attention has been given to uncountable mixtures. In the past, countable mixtures have been the primary focus of (QSD). In this work, we shall present an approach to QSD for the case of uncountable mixtures and address the respective asymptotic QSD optimization problems. We will begin with the case for countable mixtures; for a broad family of countable mixtures, we will prove necessary and sufficient conditions for minimal discrimination error to be obtained in the asymptotic regime (we say that in such a case QSD is fully solvable). Finally, we will pass to the uncountable QSD case and develop analogous results to those presented for the case of countable mixtures.

1 Introduction

Quantum State Discrimination (QSD) is the problem of minimizing the error in distinguishing between the elements of a mixture of density operators \( \sum_i p_i \hat{\rho}_i \). To understand what is meant by distinguishing we must introduce the concepts of a POVM [3].

Definition 1. POVM: Consider an arbitrary Hilbert space \( \mathcal{H} \). A trace-preserving POVM is a set of positive semi-definite operators \( \{ \hat{E}_i \} \) acting in \( \mathcal{H} \) that sum to the identity operator, i.e.
\[
\sum_i \hat{E}_i = I_{\mathcal{H}}
\]

The POVM may consist of an uncountable set of semi-definite operators as well. In such a case the analogous set of operators, e.g. \( \hat{E}_x \ (x \in \mathbb{R}) \) must meet the same constraint, i.e.
\[
\int \hat{E}_x dx = I_{\mathcal{H}}
\]

Definition 2. Quantum Measurement: Consider a POVM \( \{ \hat{E}_i \} \), acting in some Hilbert space of arbitrary dimension. Furthermore, consider a density operator \( \hat{\rho} \) which acts in the same Hilbert space. Given a quantum system in state \( \hat{\rho} \), quantum probability theory treats \( \hat{E}_i \) as events, while the traces \( p_i := Tr\{ \hat{\rho} \hat{E}_i \} \) are postulated to be the probabilities of the \( i \) the event occurring after conducting a measurement on the system designed to read out the events modeled by the POVM. We note that \( \{ \hat{E}_i \} \) have the form \( \hat{E}_i = \hat{M}_i \hat{M}_i^\dagger \). The operators \( \hat{M}_i \) known as measurement operators. If one conducts a measurement on the quantum state \( \hat{\rho} \) and the outcome is that which is indexed by \( i \), then the post-measurement state is postulated to be
\[
\frac{\hat{M}_i \hat{\rho} \hat{M}_i^\dagger}{Tr\{ \hat{M}_i \hat{\rho} \hat{M}_i^\dagger \}}
\]

The state above is the resulting state assuming that one has ‘read out’ the measurement. However, if one does not read out the results of the measurement, what one has is a mixture
\[
\sum_i p_i \frac{\hat{M}_i \hat{\rho} \hat{M}_i^\dagger}{Tr\{ \hat{M}_i \hat{\rho} \hat{M}_i^\dagger \}}
\]
Given that $p_i = \text{Tr}[\hat{E}_i\hat{\rho}]$, the unread state of the system is

$$\sum_i p_i \frac{\hat{M}_i\hat{\rho}\hat{M}_i^\dagger}{\text{Tr}\{\hat{M}_i\hat{\rho}\hat{M}_i\}} = \sum_i \frac{\text{Tr}\{\hat{M}_i\hat{\rho}\hat{M}_i\}}{\text{Tr}\{\hat{M}_i\hat{\rho}\hat{M}_i\}} = \sum_i \hat{M}_i\hat{\rho}\hat{M}_i^\dagger. \quad (5)$$

The QSD optimization problem [27] [25] [10] [5] [6] may now be defined. Let $\mathcal{H}$ be an arbitrary Hilbert space and let $\mathcal{H}$ be the space of density operators acting in $\mathcal{H}$. Given a mixture of density operators,

$$\hat{\rho} = \sum_{i=1}^N p_i\hat{\rho}_i \quad (6)$$

where $\sum_{i=1}^N p_i = 1$, the theory of QSD aims to find a POVM $\{\hat{E}_i\}_{i=1}^K \subset B(\mathcal{H})$ ($K \geq N$, $\hat{E}_i = \hat{M}_i\hat{M}_i^\dagger$) where the $\hat{M}_i$ are the corresponding measurement operators[21]) which resolves the identity operator of $B(\mathcal{H})$, and minimizes the object below which we will be referring to as a probability error.

$$p_E\{\{p_i, \hat{\rho}_i\}_{i=1}^N, \{\hat{M}_i\}_{i=1}^K\} := 1 - \sum_{i=1}^N p_i \text{Tr}\{\hat{M}_i\hat{\rho}_i\hat{M}_i^\dagger\}. \quad (7)$$

To see what is the error that (7) measures let us consider the unread measurement state (5) corresponding to the mixture $\sum_{i=1}^N \hat{\rho}_i$ after undergoing a measurement generated by the POVM $\{\hat{M}_i\hat{M}_i^\dagger\}_{i=1}^N$.

$$\sum_{j=1}^N \hat{M}_j\left(\sum_{i=1}^N p_i\hat{\rho}_i\right)\hat{M}_j^\dagger = \sum_{i=1}^N p_i\hat{M}_i\hat{\rho}_i\hat{M}_i^\dagger + \sum_{j=1}^N \sum_{i=1, i\neq j}^N p_i\hat{M}_i\hat{\rho}_i\hat{M}_j^\dagger \quad (8)$$

From the definition of POVM provided, it is clear that $\text{Tr}\{\sum_{j=1}^N \hat{M}_j\left(\sum_{i=1}^N p_i\hat{\rho}_i\right)\hat{M}_j^\dagger\} = 1$, hence from (8) and (9)

$$1 = \sum_{i=1}^N p_i \text{Tr}\{\hat{M}_i\hat{\rho}_i\hat{M}_i^\dagger\} + \sum_{j=1}^N \sum_{i=1, i\neq j}^N p_i \text{Tr}\{\hat{M}_j\hat{\rho}_i\hat{M}_j^\dagger\}. \quad (10)$$

The term $\text{Tr}\{\hat{M}_i\hat{\rho}_i\hat{M}_i^\dagger\}$ is the probability that the system modeled by the mixture (6) was in the state $\hat{\rho}_i$ given that the outcome of the measurement was the state $\hat{M}_i\left(\sum_{i=1}^N p_i\hat{\rho}_i\right)\hat{M}_i^\dagger$, meaning that $\sum_{i=1}^N p_i \text{Tr}\{\hat{M}_i\hat{\rho}_i\hat{M}_i^\dagger\}$ is the probability that POVM chosen perfectly discriminates between the different $\hat{\rho}_i$ of the mixture (6). The term $1 - \sum_{i=1}^N p_i \text{Tr}\{\hat{M}_i\hat{\rho}_i\hat{M}_i^\dagger\}$, the probability error, is hence the probability that the POVM fails to discriminate between the elements of the mixture (6). Using (10) it may also be expressed as $\sum_{j=1}^N \sum_{i=1, i\neq j}^N p_i \text{Tr}\{\hat{M}_j\hat{\rho}_i\hat{M}_j^\dagger\}$. In what follows we will just write $p_E$ in place of $p_E\{\{p_i, \hat{\rho}_i\}_{i=1}^N, \{\hat{M}_i\}_{i=1}^K\}$ as a shorthand when the context is clear. With this notation, the QSD optimization problem is the problem of computing the following minimum.

$$\min_{p_E, \hat{\rho}_i} p_E \quad (11)$$

The QSD problem will be called fully solvable when $\min_{p_E, \hat{\rho}_i} p_E = 0$.
limit as some or all of the parameters in \( \{ n_i \} \) go to infinity. We will say that the asymptotic QSD problem \textit{fully solvable} with respect to the parameter \( n_i \) when

\[
\lim_{|n_i| \to \infty} \min_{POVM} p_E \left\{ \left\{ p_i, \delta_{i,n_i}(\hat{\rho}) \right\}_{i=1}^N, \left\{ \hat{M}_i \right\}_{i=1}^K \right\} = 0
\]

the minimization above is understood to be taken for every \( n_i \).

Asymptotic QSD arises naturally in the study of quantum communication, quantum to classical transitions and quantum measurement, just to name a few applications \[12\][9][5][20]. As an example consider the case where a state is redundantly prepared by some party \( A \) in the state \( \hat{\rho}_i \) with probability \( p_i \), \( n \) copies of each state being made prior to being communicated to another party. From the perspective of some party \( B \), receiving the state prepared by \( A \), the received state would be a mixture of the following form

\[
\sum_i p_i \hat{\rho}_i^\otimes n
\]

In such a case the corresponding maps \( \delta_{i,n_i} \) have \( n_i = n \) for all \( i \) and are the map \( \hat{\rho} \) to \( \hat{\rho}^\otimes n \) for all \( i \).

Now, define \( \min_{POVM} p_E(n) := \min_{POVM} p_E \left\{ \left\{ p_i, \hat{\rho}_i^\otimes n \right\}_{i=1}^N, \left\{ \hat{M}_i \right\}_{i=1}^K \right\} \). In [22] it was shown that

\[
\frac{1}{3} C \leq - \lim_{n \to \infty} \frac{\log \left( \min_{POVM} p_E(n) \right)}{n} \leq C
\]

where \( C \) is a constant involving the quantum Chernoff bound for a mixture of \( N \) states, a more detailed discussion may be found in appendix B. Hence, for large enough \( n \), from eqn (14) we have the following inequalities.

\[
e^{-n \frac{1}{3} C} \geq \min_{POVM} p_E(n) \geq e^{-nC}
\]

This gives us an idea of how the minimum error probability drops off asymptotically as the redundancy \( n \) grows. Indeed as \( n \to \infty \) we have \( \min_{POVM} p_E(n) \to 0 \).

More recently, and more pertinently to the theme of this paper, asymptotic QSD has made an appearance in the theory of Spectrum Broadcast Structures (SBS) \[9\] for quantum measurement limit type interactions (see section 2.4 in [21] for a discussion on quantum measurement limit). The SBS framework aims at deriving a specific type of state from the dynamics; these states are called SBS states, which satisfy a notion of objectivity presented in \[9\][8][7], will emerge from the asymptotic dynamics. The definition of an SBS state stipulates the calculation of a problem related to that of QSD when proving that a state of interest converges to one of these so-called SBS states. The relevant optimization problem is now the \textit{super QSD} problem (SQSD) which is just a simple upper bound of the QSD problem. i.e. using the fact that \( \left| \text{Tr} \left\{ \hat{A} \right\} \right| \leq \| \hat{A} \|_1 \) for any trace class operator \( \hat{A} \),

\[
SQSD := \min_{POVM} \sum_{i=1}^N p_i \left\| \hat{\rho}_i - \hat{M}_i \hat{\rho}_i \hat{M}_i^\dagger \right\|_1 \geq 0
\]

\[
\min_{POVM} \sum_{i=1}^N p_i \left| \text{Tr} \left\{ \hat{\rho}_i - \hat{M}_i \hat{\rho}_i \hat{M}_i^\dagger \right\} \right| \geq 0
\]

\[
\min_{POVM} \left( 1 - \sum_{i=1}^N p_i \text{Tr} \left\{ \hat{M}_i \hat{\rho}_i \hat{M}_i^\dagger \right\} \right) = QSD
\]

In \[7\][8][9], special attention has been given to SQSD problems of the following form.

\[
\min_{POVM} \sum_i p_i \left\| e^{-it\hat{B}} \hat{\rho} e^{it\hat{B}} - \hat{M} e^{-it\hat{B}} \hat{\rho} e^{it\hat{B}} \hat{M}^\dagger \right\|_1
\]

where \( x_i \neq x_j \) for all \( i,j \neq i \) and \( \hat{B} \) is an arbitrary self-adjoint operator; the state being discriminated here is of course \( \sum_i p_i e^{-it\hat{B}} \hat{\rho} e^{it\hat{B}} \hat{B} \). Such unitarily related mixtures, with a parameter \( t \),
arise as a direct consequence of the aforementioned quantum measurement limit assumption made in [7][8][9]. The super QSD problem (19) is asymptotically fully solvable with respect to $t$ only if the associated QSD problem is fully solvable. Unlike example (13), where the asymptotic full solvability of the respective QSD problem was independent of the nature of the states involved, here this is not the case. It is easy to find examples where SQSD optimization problems of the type exhibited in (19) does not vanish as $t \to \infty$. E.g., let $\hat{B}$ be equal to the Pauli matrix $\hat{\sigma}_x$ and let $\hat{\rho} = |z_1\rangle \langle z_1|$ with $\{|z_i\rangle\}_i$ the eigenvectors of the Pauli matrix $\hat{\sigma}_x$. It is easy to show that

$$e^{-itx_1\sigma_\sigma} \hat{\rho} e^{-itx_1\sigma_\sigma} =$$

$$\cos^2(tx_1)|z_1\rangle \langle z_1| + \sin^2(tx_1)|z_2\rangle \langle z_2| +$$

$$i \cos(tx_1) \sin(tx_1)|z_1\rangle \langle z_2| - i \cos(tx_1) \sin(tx_1)|z_2\rangle \langle z_1|$$

Now consider the mixture

$$\sum_{i=1}^2 p_i e^{-itx_1\sigma_\sigma} \hat{\rho} e^{-itx_1\sigma_\sigma}$$

and let $p_i = \frac{1}{2}$ for $i = 1, 2$. An application of a result by Hellström [27], discussed in the next section leads to

$$\min_{\hat{\rho}_{\text{opt}}} p_{E}(t) = \frac{1}{2} - \frac{1}{2} \left\| e^{-itx_1\sigma_\sigma} \hat{\rho} e^{-itx_1\sigma_\sigma} - e^{-itx_1\sigma_\sigma} \hat{\rho} e^{-itx_1\sigma_\sigma} \right\|_1 =$$

$$2 \sqrt{ \left( \cos^2(tx_1) - \cos^2(tx_2) \right) \left( \sin^2(tx_1) - \sin^2(tx_2) \right) - \left( \cos(tx_1) \sin(tx_1) - \cos(tx_2) \sin(tx_2) \right)^2 }$$

Clearly (25) does not converge to zero as $t \to \infty$, ergo asymptotic QSD is not fully solvable and, by consequence of (16), neither is the associated asymptotic SQSD problem.

In this paper, we will be focusing on the QSD of unitarily related mixtures (URM); i.e. mixtures of the form $\sum_i p_i \hat{U}_i(t) \hat{\rho} \hat{U}_i(t)\dagger$, where $\hat{U}_i(t)$ are all unitary operators generated by scalar multiples of the same generator. We will provide a necessary and sufficient condition for the asymptotic full solvability of the QSD optimization problem for a broad set of URM; this condition will depend on the spectral properties of the generator of the unitary group characterizing the URM and the nature of the initial state, i.e. the state of the mixture when $t = 0$. In Sections 2, and 3 we will give an overview of some important results from the literature that we shall be using and give further motivation. In section 4 we present the main result (Theorem 8 and Corollary 2) which gives necessary and sufficient conditions for asymptotic QSD optimization of unitarily related mixtures to be fully solvable in a broad setting. In section 5 we shall introduce the optimization problem of Uncountable Quantum State Discrimination (UQSD); a framework that generalizes the problem of QSD. Drawing parallels between QSD and UQSD we prove a necessary condition for UQSD in the unitarily related mixture case to be fully solvable in the asymptotic regime with respect to a dynamical parameter $t$. This condition will again depend only on the spectral properties of the generator of the unitary group characterizing the URM and the nature of the initial state. We conclude by conjecturing that the analog of Theorem 8 is true for the UQSD case in the unitarily related mixture setting; we follow this conjecture with some motivation and intuition.

### 2 Some Important Theorems

The optimization problem (11) is, in general, be intractable; exact solutions exist only in a few special cases [5] [6]. The most famous of these cases pertains to the already mentioned so-called Hellström bound. The name may be misleading because it is not a bound. We present its statement following [5].

**Theorem 1. Hellström Bound:** Let $\mathcal{H}$ be an arbitrary Hilbert space. For any mixture of the form

$$p_1 \hat{\rho}_1 + p_2 \hat{\rho}_2 \in S(\mathcal{H}).$$

$$p_1 \hat{\rho}_1 + p_2 \hat{\rho}_2 \in S(\mathcal{H}).$$

(26)
Theorem 4. Knill and Barnum [11] of the more famous ones are the following. For any mixed quantum states the Hellström bound. Unlike the proof by Hellström for the two-state mixture, Qiu does not provide a constructive proof. We, therefore, do not have explicit knowledge of the POVM that minimizes 

Here upper bounds for the probability error in the case of a general mixture exist. Some of the more famous ones are the following. For any mixed quantum states \( \{\hat{\rho}_i\}_{i=1}^N \) with respective probabilities \( \{p_i\}_i \), Let the \( \{\hat{\rho}_i\}_i \) be density operators acting in an arbitrary Hilbert space \( \mathcal{H} \).

**Theorem 2. Qiu Bound [10]**

\[
\min_{\text{POVM}} p_E \geq \frac{1}{2} \left( 1 - \frac{1}{2} \frac{1}{(N-1)} \sum_i \sum_{j:j \neq i} \|p_i \hat{\rho}_i - p_j \hat{\rho}_j\|_1 \right)
\]

**Theorem 3. Montanaro Bound [25]:**

\[
\min_{\text{POVM}} p_E \geq \frac{1}{2} \sum_i \sum_{j:j \neq i} p_i p_j F(\hat{\rho}_i, \hat{\rho}_j)
\]

**Theorem 4. Knill and Barnum [11]**

\[
\min_{\text{POVM}} p_E \leq \sum_i \sum_{j:j \neq i} \sqrt{p_i p_j} F(\hat{\rho}_i, \hat{\rho}_j)
\]

Here \( F(\hat{\rho}, \hat{\sigma}) := \|\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}\|_1^2 \).

Theorem 4 is proven for the case of finite-dimensional Hilbert spaces. We provide a generalization of this theorem for the case where \( \mathcal{H} \) is infinite-dimensional in appendix A.

In [10], necessary and sufficient conditions are introduced in order to arrive at a generalization of the Hellström bound. Unlike the proof by Hellström for the two-state mixture, Qiu does not provide a constructive proof. We, therefore, do not have explicit knowledge of the POVM that minimizes \( p_E \). There also exist convex optimization techniques that may be employed in order to find a global minimum [5][24], but we will not be dabbling with these ideas.

We now present some results particular to the quantum fidelity.

**Theorem 5. Purification-dependent version of the Fidelity:** The quantum fidelity \( F(\hat{\rho}, \hat{\sigma}) := \|\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}\|_1^2 = \text{Tr}\left\{\sqrt{\hat{\rho} \hat{\sigma} \hat{\rho} \hat{\sigma}}\right\}^2 \) is equivalent to the following [3]:

\[
F(\hat{\rho}, \hat{\sigma}) = \max_{|\chi\rangle} |\langle \chi | \xi \rangle|^2
\]

where \( |\psi\rangle \) is any fixed purification of \( \hat{\rho} \), and the maximization is over all purifications of \( \hat{\sigma} \).

**Lemma 1. Strong Concavity of the Fidelity:** a Generalization from the equivalent theorem for singular distributions in [3]: Let \( f(x)\hat{\rho}_x dx \) and \( g(x)\hat{\sigma}_x dx \) be two uncountable mixtures (\( f(x) \) and \( g(x) \) are probability distributions). Then,

\[
\sqrt{F\left( \int f(x)\hat{\rho}_x dx, \int g(x)\hat{\sigma}_x dx \right)} \geq \sqrt{\int f(x)g(x)F(\hat{\rho}_x, \hat{\sigma}_x) dx}
\]

**Proof.** The proof follows the standard methodology, see chapter 9 of [3] for the countable mixture case. Let \( |\psi_x\rangle \) and \( |\sigma_x\rangle \) be the purifications of \( \hat{\rho}_x \) and \( \hat{\sigma}_x \) which maximize the fidelity; i.e. \( F(\hat{\rho}_x, \hat{\sigma}_x) = |\langle \psi_x | \phi_x \rangle|^2 \). We now define

\[
|\psi\rangle := \int \sqrt{f(x)} |\psi_x\rangle dx
\]

\[
|\sigma\rangle := \int \sqrt{g(x)} |\sigma_x\rangle dx
\]
Let \(|\phi\rangle := \int \sqrt{q(x)}|\phi_x\rangle|x\rangle dx\). \(|\psi\rangle\text{ and }|\phi\rangle\text{ are purifications of the operators }\int p(x)\hat{\rho}_x dx\text{ and }\int q(x)\hat{\sigma}_x dx\text{ where the ancillary space is taken to be }L^2(\mathbb{R})\). Using Theorem 5 we have.

\[
\sqrt{F\left(\int p(x)\hat{\rho}_x dx, \int q(x)\hat{\sigma}_x dx\right)} \geq |\langle \phi | \psi \rangle| = \int \sqrt{p(x)}\sqrt{q(y)}|\psi_x|\langle x | y\rangle dy dx = \int \int \sqrt{p(x)q(x)}|\psi_x|\langle x | \phi_x\rangle dx = \int \int \sqrt{p(x)q(x)}\hat{F}(\hat{\rho}_x, \hat{\sigma}_x) dx
\]

The latter gives us means by which we may bound fidelities of mixed state from below. The following corollary is immediately from Lemma 1. We present it here below.

**Corollary 1.** Let \(\int p(x)\hat{\rho}_x dx\) be an uncountable mixture, and let \(\hat{\sigma}\) be an arbitrary density operator \((p(x)\) is a probability distribution). Then,

\[
\sqrt{F\left(\int p(x)\hat{\rho}_x dx, \hat{\sigma}\right)} \geq \int p(x)\hat{F}(\hat{\rho}_x, \hat{\sigma}) dx
\]

**Proof.** Note that \(\hat{\sigma} = \int p(x)\hat{\sigma} dx\). The proof follows from applying Lemma (1) to the fidelity

\[
\sqrt{F\left(\int p(x)\hat{\rho}_x dx, \int q(x)\hat{\sigma}_x dx\right)}. \quad \square
\]

## 3 Countable Mixtures of Unitarily Related Families.

In this section, we restrict our attention to a specific type of ensemble \(\{p_i, \hat{\rho}_{i,t}\}_{i=1}^N\). Namely, let

\[
\hat{\rho}_{i,t} := e^{-it\hat{B}_i}|\psi\rangle\langle \psi|e^{it\hat{B}_i}
\]

for some self-adjoint operator \(\hat{B}\) and a pure density operator \(|\psi\rangle\langle \psi|\) both acting in an arbitrary Hilbert space \(\mathcal{H}\). The operators \(\hat{\rho}_{i,t}\) are images of the density operator \(\hat{\rho}\) under the unitary evolutions generated respectively by the operators \(x_i\hat{B}\). We will see that the asymptotic discriminability of the mixture \(\sum_{i=1}^N p_i\hat{\rho}_{i,t}\) depend on the spectral properties of the operators \(\hat{B}_k\) and on the nature of the pure state \(|\psi\rangle\). Using Theorem 4 we have the following QSD estimate.

\[
\min_{POVM} \left(1 - \sum_{i=1}^N p_i Tr\left\{\hat{M}_i\hat{\rho}_{i,t}\hat{M}_i^\dagger\right\}\right) \leq \sum_i \sum_{j: j \neq i} \sqrt{p_i p_j} \sqrt{F(\hat{\rho}_{i,t}, \hat{\rho}_{j,t})} = \sum_i \sum_{j: j \neq i} \sqrt{p_i p_j} \langle \psi | e^{-it(x_j-x_i)\hat{B}} |\psi\rangle
\]

If fully solvable asymptotic QSD is desired, then it is a necessary and sufficient condition that the elements \(|\langle \psi | e^{-it(x_j-x_i)\hat{B}} |\psi\rangle|\) of the sum above decay to zero as \(t \to \infty\) for all \(i, j; i \neq j\); it can be shown to be a necessary condition by using the bound in Theorem 3. With the latter in mind, let us first introduce some elements from spectral theory in order to understand when full solvavility of QSD is asymptotically possible for the setting at hand.

\[\text{6}\]
3.1 Spectral Decomposition and Spectral Measures.

Given a self-adjoint operator is empty [2][1]. Hence, given that \( \hat{A} \) is self-adjoint we have

\[
\text{Spec}(\hat{A}) = \text{Spec}_{ac}(\hat{A}) \cup \text{Spec}_{sc}(\hat{A}) \cup \text{Spec}_{rc}(\hat{A})
\]

where the subscripts \( ac \) and \( sc \) stand for absolutely continuous and singular continuous respectively. In order to formally define absolutely continuous and singular continuous spectra let us consider an arbitrary \( |\psi\rangle \in \mathcal{H} \); \( \mathcal{H} \) being the Hilbert space that \( \hat{A} \) acts in. The spectral theory then says that there exists a unique measure \( \mu_{\psi} \) such that [1]

\[
\langle \psi | \hat{A} | \psi \rangle = \int_{\mathbb{R}} \lambda d\mu_{\psi}(\lambda).
\]

The measure \( \mu_{\psi} \) is often called the spectral measure associated with \( |\psi\rangle \). By the Lebesgue Decomposition Theorem one may decompose any measure of this type into its point measure, absolutely continuous measure, and singular continuous measure components. i.e.

\[
\mu_{\psi} = \mu_{\psi,p} + \mu_{\psi,ac} + \mu_{\psi,sc}.
\]

Of particular interest to us will be the properties of the Fourier transforms of each of the measures on the right-hand side of (44). It is a consequence of the Riemann-Lebesgue Lemma that the Fourier transform of \( \mu_{\psi,ac} \) (absolutely continuous with respect to the Lebesgue measure) is a function that decays to zero as the argument becomes large. On the other hand, it can be shown that the Fourier transform of \( \mu_{\psi,sc} \) does not decay to zero, however, it does in some cases. E.g. Fourier transform of (e.g. Cantor distribution (devils staircase), Dirac measure) is known not to decay to zero in general; However, there exist singular continuous measures whose Fourier transforms decay to zero Fourier transforms of absolutely continuous measures. We shall be particularly interested in the subset of measures continuous with respect to the Lebesgue measures whose Fourier transform decays to zero. These will provide the necessary dynamics for the bound (4) to converge to zero for the case of the mixture presented above, i.e. \( \sum_{i=1}^{N} p_i \hat{\rho}_i \).

Let \( \hat{A} \) be a self-adjoint operator acting in a Hilbert space \( \mathcal{H} \). \( \mathcal{H} \) may furthermore be expressed as a direct sum of three invariant subspaces; one corresponding to each type of spectrum. Namely, from [2]

\[
\mathcal{H} = \mathcal{H}_p \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sc}
\]

Recall that the QSD problem may be fully solved asymptotically iff \( \forall i \neq j \)

\[
|\langle \psi | e^{-it(x_j-x_i)} \hat{B} | \psi \rangle| \to 0 \quad (as \quad t \to \infty)
\]

Using the spectral theorem for unitary operators [1] it immediately follows that (46) may be written as

\[
|\langle \psi | e^{-it(x_j-x_i)} \hat{B} | \psi \rangle| = \left| \int_{\mathbb{R}} e^{-it(x_j-x_i)\lambda} d\mu_{\psi}(\lambda) \right| \leq \int_{\mathbb{R}} e^{-it(x_j-x_i)\lambda} d\mu_{\psi,p}(\lambda) + \int_{\mathbb{R}} e^{-it(x_j-x_i)\lambda} d\mu_{\psi,ac}(\lambda) + \int_{\mathbb{R}} e^{-it(x_j-x_i)\lambda} d\mu_{\psi,sc}(\lambda)
\]

It is now clear why we are interested in the Fourier transforms of the \( p, ac \) and \( sc \) measures of the operator \( \hat{B} \). If we expect \( |\langle \psi | e^{-it(x_j-x_i)} \hat{B} | \psi \rangle| \to 0 \) as \( t \to \infty \), then we know that it will be necessary (but not sufficient!) for \( |\psi\rangle \in \mathcal{H}_{ac} \cup \mathcal{H}_{sc} \). \( |\psi\rangle \in \mathcal{H}_{ac} \cup \mathcal{H}_{sc} \) implies the desired decay but the same is not true about \( \mathcal{H}_{sc} \); everything in \( \mathcal{H}_{ac} \) will yield the dynamics we want but not everything in \( \mathcal{H}_{sc} \). We must hence constrain ourselves further to the subspace \( \mathcal{H}_{rc} \) consisting only of the states \( |\psi\rangle \) whose associated measure \( \mu_{\psi} \) is a Rajchman measure [19] (defined below). The associated invariant subspace is exactly what we need.
Definition 3. Rajchman Measure: A finite Borel probability measure \( \mu \) on \( \mathbb{R} \) is called a Rajchman measure if it satisfies
\[
\lim_{t \to \infty} \hat{\mu}(t) = 0
\]
where \( \hat{\mu}(t) := \int_{\mathbb{R}} e^{2\pi itx} d\mu(x) \), \( t \in \mathbb{R} \).

Theorem 6. Let \( \hat{A} \) be a self-adjoint operator acting on some arbitrary Hilbert space \( \mathcal{H} \), then the set of vectors in \( \mathcal{H} \) for which the spectral measure is a Rajchman measure, i.e.
\[
\mathcal{H}_{rc} := \{ |\psi\rangle | \lim_{t \to \infty} \langle \psi | e^{-it\hat{A}} |\psi\rangle = 0 \},
\]
is a closed subspace which is invariant under \( e^{-it\hat{A}} \) [19].

Lemma 2. If \( \mu_{\psi} \) is Rajchman, then \( \mu_{\phi,\psi} \) is Rajchman: Let \( \hat{B} \) be some self-adjoint operator acting on a Hilbert space \( \mathcal{H} \). Furthermore, let \( |\psi\rangle \in \mathcal{H}_{rc} \) and \( |\phi\rangle \in \mathcal{H} \), then the respective measure \( \mu_{\phi,\psi} \) is Rajchman.

Proof.\[
\int e^{-it\lambda} \mu_{\phi,\psi}(\lambda) = \langle \phi | e^{-it\hat{B}} |\psi\rangle = \langle \phi | \hat{P}_{rc} e^{-it\hat{B}} |\psi\rangle = \langle \xi | e^{-it\hat{B}} |\psi\rangle
\]
where \( \hat{P}_{rc} \) is the projector onto the subspace \( \mathcal{H}_{rc} \) and \( |\xi\rangle := \hat{P}_{rc} |\phi\rangle \in \mathcal{H}_{rc} \). We have used the fact that the Rajchman subspace is invariant under the action of \( e^{-it\hat{B}} \). Now, using the polarization identity (see [13] chapter 2 Exercise 2.1) we have
\[
\langle \xi | e^{-it\hat{B}} |\psi\rangle = \frac{1}{4} \sum_{k=0}^{3} i^{k} \left( \langle \xi | + (-i)^{k} \langle \psi | \right) e^{-it\hat{B}} \left( |\xi\rangle + i^{k} |\psi\rangle \right)
\]
where we have defined \( |\chi_{k}\rangle : |\xi\rangle + i^{k} |\psi\rangle \). \( \mathcal{H}_{rc} \) is a linear space, hence \( |\chi_{k}\rangle \in \mathcal{H}_{rc} \) for \( k = 0, 1, 2, 3 \).

Piecing all together.
\[
\int e^{-it\lambda} \mu_{\phi,\psi}(\lambda) = \frac{1}{4} \sum_{k=0}^{3} i^{k} \langle \chi_{k} | e^{-it\hat{B}} |\chi_{k}\rangle = \frac{1}{4} \sum_{k=0}^{3} i^{k} \int e^{-it\lambda} d\mu_{\chi_{k}}(\lambda).
\]
As \( t \to \infty \) \( \int e^{-it\lambda} d\mu_{\chi_{k}}(\lambda) \to 0 \). Hence \( \int e^{-it\lambda} \mu_{\phi,\psi}(\lambda) \to 0 \) as \( t \to \infty \). \( \square \)

We conclude this subsection with the following proposition.

Proposition 1. Full Solvability of QSD for URM of Pure States: Consider the model described in this section by the states (39). \( |\psi\rangle \in \mathcal{H}_{rc} \) corresponding to \( \hat{B} \) iff
\[
\lim_{t \to \infty} \min_{POVM} \{ p_{i}, e^{-it\hat{B}} |\psi\rangle \langle \psi | e^{it\hat{B}} \}_{i=1}^{N}, \{ \hat{M}_{l} \}_{l=1}^{K} \} = 0
\]
Proof. This immediately follows from Theorems 4 and 3. \( \square \)
4 Unitarily Related Mixtures of Finite Mixtures.

Let us now consider the case where

\[
\hat{\rho} = \sum_{i=1}^{N} p_i \hat{\rho}_i \in \mathcal{S}(\mathcal{H}),
\]

with

\[
\hat{\rho}_i = \sum_{j=1}^{M} \eta_{ij} |\phi_{ij}\rangle \langle \phi_{ij}| \tag{58}
\]

with all of the $|\phi_{ij}\rangle \in \mathcal{H}$ of norm one and $\sum_j \eta_{ij} = 1$. In such a case we may again utilize Theorem 4 to begin with.

\[
\min_{POVM} p_E(t) \leq \sum_{i,j \neq i} \sqrt{p_i p_j} \sqrt{F(\hat{\rho}_i, \hat{\rho}_j)} \tag{59}
\]

However, the fidelities in this case are not immediately manageable owing to the fact that both $\hat{\rho}_i$ and $\hat{\rho}_j$ are both mixed states. To overcome this hurdle we will use a bound for quantum fidelities found in [23]. Namely,

**Theorem 7.** Fidelity Bound Koenraad and Milan [23]: Let $\sum_i p_i \hat{\rho}_i$ be an arbitrary countable mixture and let $\hat{\sigma}$ be an arbitrary density operator; both acting on the same arbitrary Hilbert space. Then,

\[
\sqrt{F\left(\sum_i p_i \hat{\rho}_i, \hat{\sigma}\right)} \leq \sum_i \sqrt{p_i} \sqrt{F(\hat{\rho}_i, \hat{\sigma})} \tag{60}
\]

Note that this theorem for the general case of an infinite mixture would require that $\sum \sqrt{p_i} \infty$.

Applying Theorem 7 twice we may further bound (59) to obtain

\[
\min_{POVM} p_E \leq \sum_{i,j \neq i} \sqrt{p_i p_j} \sum_{k=1}^{M} \sum_{k'=1}^{M} \sqrt{\eta_{ik} \eta_{ik'}} \sqrt{F(|\phi_{ik}\rangle \langle \phi_{ik}|, |\phi_{ik'}\rangle \langle \phi_{ik'}|)} = \tag{61}
\]

\[
\sum_{i,j \neq i} \sqrt{p_i p_j} \sum_{k=1}^{M} \sum_{k'=1}^{M} \sqrt{\eta_{ik} \eta_{ik'}} \left| \langle \phi_{ik} | \phi_{ik'} \rangle \right| \tag{62}
\]

We hence see that the optimal probability error may be controlled by the inner products $\left| \langle \phi_{ik} | \phi_{ik'} \rangle \right|$ ($i \neq j$), which are relatively easy to compute. We now provide a generalization of Proposition 1.

**Theorem 8.** Full Solvability of QSD for URM of Finite Mixtures: Let $\mathcal{H}$ be infinite-dimensional Hilbert space. Let $\hat{B}$ be a self-adjoint operator acting in $\mathcal{H}$ with a non-empty Rajchman subspace. Furthermore, let $\hat{\rho}_i := \sum_{j=1}^{M_i} \eta_{ij} |\phi_{ij}\rangle \langle \phi_{ij}|$ be finite mixtures in $\mathcal{S}(\mathcal{H})$ for each $i$. Then,

\[
\lim_{t \to \infty} \min_{POVM} p_E \left\{ \left\{ p_i, e^{-itx_i} \hat{B} \hat{\rho}_i e^{itx_i} \hat{B} \right\}^N \right\} = 0 \tag{63}
\]

iff all of the $|\phi_{ij}\rangle \in \mathcal{H}_{rc}$ of $\hat{B}$.

**Proof.** First we assume that $|\phi_{ij}\rangle \in \mathcal{H}_{rc}$ of $\hat{B}$ for all $ij$. Now, using (62) we have

\[
\min_{POVM} p_E(t) \leq \sum_{i,j \neq i} \sqrt{p_i p_j} \sum_{k=1}^{M_i} \sum_{k'=1}^{M_j} \sqrt{\eta_{ik} \eta_{jk'}} \left| \langle \phi_{ik} | e^{-it(x_j-x_i)} \hat{B} | \phi_{jk} \rangle \right| \tag{64}
\]

Since all of the sums above are finite, we need only worry about the limits

\[
\lim_{t \to \infty} \left| \langle \phi_{ik} | e^{-it(x_j-x_i)} \hat{B} | \phi_{jk} \rangle \right| \tag{65}
\]
but by Lemma 2 these all go to zero as \( t \to \infty \). We have therefore proven one direction of the theorem.

Going the other way we shall prove the contrapositive. Assume that \( |\phi_{ij}| \notin \mathcal{H}_c \) for all \( ij \). Using Theorem 3 we have

\[
\min_{\text{POVM}} \rho_p(t) \geq \frac{1}{2} \sum_i \sum_{j:j \neq i} p_i p_j F \left( e^{-tx, \hat{B}} \hat{p}_i e^{tx, \hat{B}}, e^{-tx, \hat{B}} \hat{p}_j e^{tx, \hat{B}} \right) \geq \sum_i \sum_{j:j \neq i} p_i p_j \left( \min \left\{ M_i, M_j \right\} \right) \frac{\lambda}{\sqrt{\eta_i \eta_j}} \left| \langle \phi_{ik} | e^{-it(x_j - x_i) \hat{B}} | \phi_{jk} \rangle \right|^2 \tag{66}
\]

In this case the terms \( \left| \langle \phi_{ik} | e^{-it(x_j - x_i) \hat{B}} | \phi_{jk} \rangle \right|^2 \) will be bounded away from zero infinitely often. Making \( \min_{\text{POVM}} \rho_p(t) \) bounded away from zero infinitely often. Hence, asymptotic QSD is impossible. The theorem has been proved.

\[\square\]

Corollary 2. QSD 2 with \( \sum_j \sqrt{\eta_j} < \infty \) for all \( j \): Theorem 8 may be extended to the cases where the finite mixtures \( \hat{p}_i \) are replaced by infinite mixtures \( \hat{p}_i := \sum_{j=1}^{\infty} \eta_{ij} |\phi_{ij}\rangle \langle \phi_{ij}| \), where now \( \sum_{j=1}^{\infty} \eta_{ij} = 1 \) for all \( i \), if \( \sum_j \sqrt{\eta_j} < \infty \) for all \( i \). The argument follows by applying the dominated convergence theorem to the first part of our proof for Theorem 8.

Corollary (2) gives us a way to work with the spectral decomposition of the operators in the mixtures \( \sum_i p_i e^{-itx, \hat{B}} \hat{p}_i e^{itx, \hat{B}} \), so long as the sequence \( \sqrt{\eta_j} \) of square-rooted eigenvalues of each \( \hat{p}_i \) is summable with respect to \( j \).

## 5 Uncountable Mixtures.

Consider the case where instead of a countable mixture, as seen in (6), we have an uncountable one.

\[\hat{p}_1 := \int p(x) \hat{p}_{x,t} dx \tag{68}\]

where \( \hat{p}_{x,t} := e^{-itx \hat{B}} |\psi\rangle \langle \psi| e^{itx \hat{B}}, |\psi\rangle \langle \psi| \) an initial state in \( \mathcal{S}(\mathcal{H}) \), \( \mathcal{H} \) an infinite dimensional Hilbert space, \( \hat{B} \) a self-adjoint operator acting in \( \mathcal{H} \) and \( \int p(x)dx = 1 \). The states \( \hat{p}_{x,t} \) are akin to the archetypal ensembles which are the main focus of QSD. In the literature \([5][10][25][11]\) for QSD, one almost always encounters ensembles of the form \( \sum_i p_i \hat{p}_i \) ( \( p_i \) is a discrete probability distribution) and the task is to find a POVM that minimizes \( \sum_i p_i \text{Tr} \left( \hat{M}_i - \hat{M}_i \hat{M}_i^\dagger \right) \) while satisfying \( \sum_i \hat{M}_i^\dagger \hat{M}_i = 1 \). If we wanted to discriminate between the \( \hat{p}_{x,t} \) with high precision, we would expect that \( F(\hat{p}_{x,t}, \hat{p}_{y,t}) \) (where \( F(,) \) is the quantum fidelity defined in Theorem 5) should go to zero as \( t \to \infty \) for all \( x \neq y \).

To see that the latter is not the case in general, recall that \( F(\hat{p}_{x,t}, \hat{p}_{y,t}) = \langle \psi | e^{-it(y-x)\hat{B}} | \psi \rangle \). Indeed, for fixed \( x \neq y \), \( \langle \psi | e^{-it(y-x)\hat{B}} | \psi \rangle \to 0 \) as \( t \to \infty \) whenever \( |\psi\rangle \in \mathcal{H}_c \) (Rajchman subspace associated with \( \hat{B} \), see Section 4). However, if for every \( t \) we choose \( y \) and \( x \) so that \( x - y = \frac{\pi}{\alpha} \), then \( F(\hat{p}_{x,t}, \hat{p}_{y,t}) = \langle \psi | e^{-i\alpha \hat{B}} | \psi \rangle \). If \( \alpha \) is small, then \( F(\hat{p}_{x,t}, \hat{p}_{y,t}) \) may be close to one. We therefore abandon the idea of discriminating all of the \( \hat{p}_{x,t} \) from one another and will instead rely on the already existing theory of QSD for countable mixtures. We will do this by defining a concept of \( N \)-mixture associated with the uncountable mixture (68). To motivate the latter we first consider a partition of the support of \( p(x) \) into \( N \) intervals: \( \bigcup_{i=1}^{N} \Omega_i = \text{supp}(p(x)) \). Using this partition we may rewrite (68) as follows.

\[\int p(x) \hat{p}_x dx = \sum_{i=1}^{N} \int_{\Omega_i} p(x) \hat{p}_{x,t} dx. \tag{69}\]

Next we define a discrete probability distribution \( p_i := \int_{\Omega_i} p(x) dx \) and \( \tilde{p}(x) := \frac{p(x)}{p_i} \) and rewrite (69)

\[\sum_{i=1}^{N} \int_{\Omega_i} p(x) \hat{p}_{x,t} dx = \sum_{i=1}^{N} p_i A_{i,t} (|\psi\rangle \langle \psi|) \tag{70}\]
Here \( \Lambda_{i,t}(\psi\langle\psi|) := \int_{\Omega_i} \overline{\rho(x)e^{-i t x \mathbf{\hat{B}}}}\psi\langle\psi|e^{i t x \mathbf{\hat{B}}} dx \), so that \( \Lambda_{i,t}(\psi\langle\psi|) \) is a density operator (not pure in general). The fact that the \( t = 0 \) state was assumed to be pure was not essential, so in the following definition we replace it by a general state. Let us now formally define an \( N \)-mixture corresponding to a given uncountable mixture.

**Definition 4. N-mixture:** Let \( \hat{\rho} := \int p(x)\hat{\rho}_{x,t} dx \) be an uncountable mixture. We call the following an \( N \)-mixture of \( \hat{\rho} \) with respect to some partition \( \cup_{i=1}^{N} \Omega_i \) (of \( N \) elements) of the support of \( p(x) \).

\[
\sum_{i=1}^{N} p_i \hat{\rho}_{i,t} \tag{71}
\]

where \( p_i := \int_{\Omega_i} p(x) dx \), \( \bar{p}(x) := \frac{p(x)}{p(t)} \) and \( \hat{\rho}_{i,t} := \int_{\Omega_i} \bar{p}(x)\hat{\rho}_{x,t} dx \). We emphasize that this is not an approximation but merely a way of rewriting \( \hat{\rho} \); also note that the \( \hat{\rho}_{i,t} \) are density operators.

Given the mixture (71), we can use the theory of countable mixture QSD in order to estimate an optimal POVM that minimizes \( \sum_i p_i Tr\{\hat{\rho}_{i,t} - \hat{M}_{i,t}\hat{M}_{i,t}^\dagger\} \) (in this case \( \hat{\rho}_{i,t} := \int_{\Omega_i} \bar{p}(x)\hat{\rho}_{x,t} dx \), and in the case where finding the minimizing POVM is not possible we may bound the min error by making use of the Knill Barnum bound (30) [11] in order to study the theoretical effectiveness of the related QSD problem with respect to \( t \), i.e. we would like to know if the associated QSD problem is fully solvable with respect to \( t \) or not. We now formalize the QSD problem for uncountable mixtures (UQSD).

**Definition 5. QSD for uncountable mixtures (UQSD):** Let \( \mathcal{H} \) be an arbitrary Hilbert space. Now, consider the unaccountably mixed state \( \hat{\rho}_i := \int p(x)e^{-i t x \mathbf{\hat{B}}}\hat{\rho}e^{i t x \mathbf{\hat{B}}} dx \), \( p(x) \) a probability density, \( \hat{\rho} \in \mathcal{S}(\mathcal{H}) \) some initial state, and \( \mathbf{\hat{B}} \) a self adjoint operator (acting in \( \mathcal{H} \)). Furthermore, consider an \( N \)-mixture of \( \hat{\rho}_i \) with respect to some partition of the support of \( p(x) \), \( \cup_{i=1}^{N} \Omega_i \) (\( N \) elements). We call the associated optimization problem below the UQSD problem induced by the partition \( \cup_{i=1}^{N} \Omega_i \).

\[
\min_{POVM} \sum_{i=1}^{N} p_i \left( 1 - Tr\{\hat{M}_i \hat{\rho}_{i,t} \hat{M}_i^\dagger\} \right) \tag{72}
\]

where now \( p_i := \int_{\Omega_i} p(x) dx \), \( \bar{p}_i := \frac{p(x)}{p(t)} \) and \( \hat{\rho}_{i,t} := \int_{\Omega_i} \bar{p}_i(x)e^{-i t x \mathbf{\hat{B}}}\hat{\rho}e^{i t x \mathbf{\hat{B}}} dx \).

An uncountable number of \( N \)-mixtures may be generated for any given uncountable mixture. If no constraints on the magnitudes of the \( \Omega_i \) are posed, trivial \( N \) mixtures might be devised. e.g. consider the case of a 2-Mixture for some uncountable mixture \( \int p(x)\hat{\rho}_{x,t} dx \). For every \( \varepsilon > 0 \) we may choose \( \Omega_2 \) such that \( ||\int_{\Omega_1} \bar{p}(x)\hat{\rho}_{x,t} dx||_1 = \varepsilon \). Consequently \( ||\int_{\Omega_1} \bar{p}(x)\hat{\rho}_{x,t} dx||_1 \geq 1 - \varepsilon \) where \( \Omega_1 = \text{supp}\{p(x)\} - \Omega_2 \). Using the Hellström bound we get the following result.

\[
\min_{POVM} p_E \left\{ \left\{ p_i, \int_{\Omega_i} p(x)\hat{\rho}_{x,t} dx \right\}^{2}_{i=1}, \left\{ \hat{M}_i \right\}^{2}_{i=1} \right\} = \tag{73}
\]

\[
\frac{1}{2} - \frac{1}{2} \left\| \int_{\Omega_1} p(x)\hat{\rho}_{x,t} dx - \int_{\Omega_2} p(x)\hat{\rho}_{x,t} dx \right\|_1 \leq \tag{74}
\]

\[
\frac{1}{2} - \frac{1}{2} \left\| \int_{\Omega_1} p(x)\hat{\rho}_{x,t} dx \right\|_1 - \left\| \int_{\Omega_2} p(x)\hat{\rho}_{x,t} dx \right\|_1 = \tag{75}
\]

\[
\frac{1}{2} - \frac{1}{2} \left\| \int_{\Omega_1} p(x)\hat{\rho}_{x,t} dx \right\|_1 - \varepsilon \leq \frac{1}{2} - \frac{1}{2} \left| 1 - 2\varepsilon \right| \approx 0 \tag{76}
\]

We will hence only be interested in the UQSD optimization problem induced by partitions \( \cup_{i=1}^{N} \Omega_i \) with magnitude constraints, i.e. \( \Delta_{i,L} \leq ||\Omega_i|| \leq \Delta_{i,U} \) for every \( i \), e.g. \( ||\Omega_i|| > \Delta \) for all \( i \). Which \( N \)-mixtures are physical and which are not is a question that has not been addressed as of yet and would be and be a problem best addressed through the lens of Quantum Metrology [1], a topic which is beyond the scope of this work.
Now that we have defined QSD for uncountable mixtures, we may ask ourselves if an adaptation of Proposition 1 is possible for this setting. Morally speaking this must be so; however, due to the intractability of the fidelity $F(\hat{\rho}, \hat{\sigma})$ for the case where both $\hat{\rho}$ and $\hat{\sigma}$ are not pure states, the argument is not as direct as it was in Proposition 1 and owing to the uncountably mixed nature of the operators $\hat{\rho}_{t,t}$ from (72) we may not apply the techniques all of the techniques used in proving Theorem 8. In fact, we may only prove that the initial state $\hat{\rho} \in \mathcal{S}(\mathcal{H}_c)$ of $\hat{B}$ is a necessary condition for the associated asymptotic UQSD problem to be fully solvable. We prove the latter result for UQSD and then present a conjecture.

Proposition 2. Necessary Condition for Full Solvability of UQSD for URM: Consider the setup of Definition 5. For the UQSD optimization problem induced by some partition $\cup_{i=1}^{N} \Omega_i$ to be fully solvable as $t \to \infty$, $\hat{\rho} \in \mathcal{S}(\mathcal{H}_c)$ (Rajchman subspace of the operator $\hat{B}$) is a necessary condition.

Proof. First, let us consider the spectral decomposition of the initial state $\hat{\rho}$. Namely,

$$\hat{\rho} = \sum_{i} \lambda_i |\psi_i\rangle \langle \psi_i|$$

Because $\hat{\rho} \in \mathcal{S}(\mathcal{H}_c)$ it is the case that the $|\psi_i\rangle$ are all in $\mathcal{H}_c$. Now, using Theorem 3 followed by Corollary 1 two times, subsequently followed by an application of Theorem 1 we see that

$$\min_{\text{POVM}} \sum_{i=1}^{N} \rho_i \left(1 - \text{Tr} \{\hat{M}_i \hat{\rho}_{t,t} \hat{M}_i^\dagger\}\right) \geq \frac{1}{2} \sum_{i,j \neq i} p_i p_j F(\hat{\rho}_{t,t}, \hat{\rho}_{t,t}) \geq$$

$$\frac{1}{2} \sum_{i,j \neq i} p_i p_j \left( \int_{\Omega_i} \bar{p}_i(x) \left( \int_{\Omega_j} \bar{p}_j(y) F(e^{-itx} \hat{B} \rho e^{itx} \hat{B}, e^{-ity} \hat{B} \rho e^{ity} \hat{B}) dy \right)^2 dx \right)^2 \geq$$

$$\frac{1}{2} \sum_{i,j \neq i} p_i p_j \left( \int_{\Omega_i} \bar{p}_i(x) \left( \int_{\Omega_j} \bar{p}_j(y) \left( \sum_{i} \lambda_i |\psi_i\rangle \langle \psi_i| e^{-itx} \hat{B} \rho e^{-itx} \hat{B} \rangle \langle \psi_i| e^{-ity} \hat{B} \rho e^{-ity} \hat{B} \rangle \right)^2 dy \right)^2 dx \right)^2 \geq$$

$$\frac{1}{2} \sum_{i,j \neq i} p_i p_j \left( \int_{\Omega_i} \bar{p}_i(x) \left( \int_{\Omega_j} \bar{p}_j(y) \left( \sum_{i} \lambda_i |\psi_i\rangle \langle \psi_i| e^{-it(x-y)} \hat{B} \rho e^{-it(x-y)} \hat{B} \rangle \langle \psi_i| e^{-ity} \hat{B} \rho e^{-ity} \hat{B} \rangle \right)^2 dy \right)^2 dx \right)^2 \geq$$

$$\frac{1}{2} \sum_{i,j \neq i} p_i p_j \left( \int_{\Omega_i} \bar{p}_i(x) \left( \int_{\Omega_j} \bar{p}_j(y) \left( \sum_{i} \lambda_i |\psi_i\rangle \langle \psi_i| e^{-it(x-y)} \hat{B} \rho e^{-it(x-y)} \hat{B} \rangle \langle \psi_i| e^{-ity} \hat{B} \rho e^{-ity} \hat{B} \rangle \right)^2 dy \right)^2 \right)^2$$

If $|\psi_i\rangle \notin \mathcal{H}_c$ for all $i$, then $\langle \psi_i \rangle e^{-it(x-y)} \hat{B} \rho e^{-it(x-y)} \hat{B} \rangle \langle \psi_i \rangle$ will not decay to zero for $x \neq y$; this behaviour is necessary for the lower bound (81) to go to zero. We have therefore proven the necessity of the condition $|\psi_i\rangle \in \mathcal{H}_c$ and hence $\hat{\rho} \in \mathcal{S}(\mathcal{H}_c)$; more than this we have also given a lower bound which may be used to estimate how much UQSD problems deviates from 0.

As mentioned already in this work, it is unknown whether or not the hypothesis of Proposition 2 is sufficient to guarantee that an UQSD problem induced by some partition $\cup_{i=1}^{N} \Omega_i$ is asymptotically fully solvable. We formally conjecture that this is the case below.

Conjecture 1. iff for Proposition 2: Assuming the same definitions used hitherto for the following symbols. The UQSD optimization problem induced by some partition $\cup_{i=1}^{N} \Omega_i$ is asymptotically fully solvable iff $\hat{\rho} \in \mathcal{S}(\mathcal{H}_c)$ of the operator $\hat{B}$.

As a finishing note, we motivate Conjecture 1. For this, we will need the super fidelity.

Theorem 9. Super Fidelity [26]: For any two density operators $\hat{\rho}$ and $\hat{\sigma}$, then

$$F(\hat{\rho}, \hat{\sigma}) \leq \text{Tr} \{\hat{\rho} \hat{\sigma}\} + \sqrt{(1 - \text{Tr} \{\hat{\rho}^2\})(1 - \text{Tr} \{\hat{\sigma}^2\})}$$

(82)
Let us now consider the uncountable unitarily related mixture $\int p(x)e^{-itx\hat{B}}\langle \psi | e^{itx\hat{B}} dx$ where $|\psi\rangle \in \mathcal{H}_{rc}$ of $\hat{B}$. Let $p(x)$ be a bimodal probability density $p(x) = \frac{1}{2}(p_1(x) + p_2(x))$ of two probability densities with non-overlapping compact support. Let $\Delta_1 \subset \mathbb{R}$ and $\Delta_2 \subset \mathbb{R}$ be their supports respectively. Now, for $\Delta_1$ and $\Delta_2$ with any magnitude, i.e. $\delta_1 := |\Delta_1|$ and $\delta_2 := |\Delta_2|$ and for any $\varepsilon_1 > 0$ we may find a time domain $\mathcal{T} := [0,T]$ so that

$$Tr\left\{ \left( \int p_i(x)e^{-itx\hat{B}}\langle \psi | e^{itx\hat{B}} dx \right)^2 \right\} \geq 1 - \varepsilon$$

for all $t \in \mathcal{T}$ and $i = 1, 2$. Furthermore, with $\mathcal{T}$ fixed, for any $\varepsilon_2 > 0$ we can choose $\text{dist}(\Delta_1, \Delta_2)$ such that

$$Tr\left\{ \int p_1(x)e^{-itx\hat{B}}\langle \psi | e^{itx\hat{B}} dx \right\} < \varepsilon_2$$

Proof. Fix $\varepsilon_2 > 0$ and let $t' \in \mathcal{T}$. Now,

$$Tr\left\{ \int p_1(x)e^{-itx\hat{B}}\langle \psi | e^{itx\hat{B}} dx \right\} = \int \int p_1(x)p_2(x)\langle \psi | e^{-it(y-x)\hat{B}}\psi \rangle^2 dxdy$$

for all $t \in [t', T]$.

With the latter and the use of Theorem 9 we have the following result.

$$F\left( \int p_1(x)e^{-itx\hat{B}}\langle \psi | e^{itx\hat{B}} dx, \int p_2(x)e^{-itx\hat{B}}\langle \psi | e^{itx\hat{B}} dx \right) \leq \varepsilon_2 + \varepsilon_1$$

meaning that we may approximately solve the UQSD problem for the 2-mixture $\sum_{i=1}^2 p_i\hat{\rho}_i$ where $p_1 = p_2 = \frac{1}{2}$ and of course $\hat{\rho}_i := \int p_i(x)e^{-itx\hat{B}}\langle \psi | e^{itx\hat{B}} dx$. We can proceed similarly for a multimode probability density. We can always place the lumps far enough apart from one order in such a way that we may observe decay in the optimal probability of error long before we see an error in our bound due to our super fidelity estimate estimates. It is clear that $|\psi\rangle \in \mathcal{H}_{rc}$ of $\hat{B}$ plays a key role in going from (86 to (87); without this assumption our conclusion is not attainable.

A Proof of the Knill-Barnum for infinite-dimensional Density Operators.

When (30) was proven in [11], the assumption was that the underlying Hilbert space was finite-dimensional. The inversion of the mixture was used as a crucial step in the proof; an operation that can not be implemented on infinite dimensional compact operators. It is worthwhile to reassure oneself that the Knill and Barnum bound may be extended to the case where the underlying Hilbert space is infinite-dimensional. We begin with a lemma and then proceed to prove the generalization. Consider the mixture $\sum p_i\hat{\rho}_i$ of density operators, all acting in an infinite dimensional Hilbert space ($\sum p_i = 1$).

Lemma 3. Let $\hat{\rho}_{i,d} = \sum_{k=1}^d \lambda_k |\psi_{ki}\rangle\langle \psi_{ki}|$ be a rank $d$ approximation of the operator $\hat{\rho}_i$. Then

$$\lim_{d \to \infty} \| \sqrt{\hat{\rho}_{i,d}} \sqrt{\hat{\rho}_{j,d}} \|_1 = \| \sqrt{\hat{\rho}_i} \sqrt{\hat{\rho}_j} \|_1$$

(89)
Proof.

\[
\lim_{d \to \infty} \left\| \sqrt{\hat{\rho}_{i,d}} \sqrt{\hat{\rho}_{j,d}} \right\|_1 = \left\| \sqrt{\hat{\rho}_i} \sqrt{\hat{\rho}_j} \right\|_1 \leq \quad \quad (90)
\]

\[
\lim_{d \to \infty} \left\| \sqrt{\hat{\rho}_{i,d}} \sqrt{\hat{\rho}_{j,d}} - \sqrt{\hat{\rho}_i} \sqrt{\hat{\rho}_j} \right\|_1 \leq \quad \quad (91)
\]

\[
\lim_{d \to \infty} \left\| \sqrt{\hat{\rho}_{i,d}} - \sqrt{\hat{\rho}_i} \right\|_2 \left\| \sqrt{\hat{\rho}_{j,d}} \right\|_2 + \lim_{d \to \infty} \left\| \sqrt{\hat{\rho}_j} - \sqrt{\hat{\rho}_i} \right\|_2 \left\| \sqrt{\hat{\rho}_j} \right\|_2 \leq \quad \quad (92)
\]

\[
\lim_{d \to \infty} \left\| \sqrt{\hat{\rho}_{i,d}} - \sqrt{\hat{\rho}_i} \right\|_2 = \quad \quad (93)
\]

\[
\lim_{d \to \infty} \left\| \sum_{k=d+1}^{\infty} \sqrt{\lambda_{k,i} |\psi_{ki}\rangle \langle \psi_{ki}|} \right\|_2 + \lim_{d \to \infty} \left\| \sum_{k=d+1}^{\infty} \sqrt{\lambda_{k,j} |\psi_{kj}\rangle \langle \psi_{kj}|} \right\|_2 = \quad \quad (94)
\]

\[
\lim_{d \to \infty} \left( \sum_{k=d+1}^{\infty} \lambda_{k,i} + \sum_{k=d+1}^{\infty} \lambda_{k,j} \right) = \lim_{d \to \infty} \sum_{k=d+1}^{\infty} \left( \sqrt{\lambda_{k,i}} + \sqrt{\lambda_{k,j}} \right) = 0. \quad \quad (95)
\]

We are now ready to generalize Theorem 4

**Theorem 10.** Knill and Barnum bound extended to infinite dimensional density operators. Let \( \sum_{i=1}^{N} p_i \hat{\rho}_i \) be some finite mixture of infinite dimensional density operators, then the Knill Barnum (30) bound applies to such a mixture.

**Proof.** Starting from the optimization problem \( \min_{POVM} \sum_{i=1}^{N} \sum_{j \neq i}^{M} p_i Tr(\hat{M}_j \hat{\rho}_i \hat{M}_j^\dagger) \), notice that we may rewrite each \( \hat{\rho}_i \) as the limit of a sequence of finite rank operators. To see this, first we diagonalize the \( \hat{\rho}_i \), i.e. \( \hat{\rho}_i = \sum_{k=1}^{\infty} \lambda_{k,i} |\psi_{ki}\rangle \langle \psi_{ki}| \). Where the \( \lambda_{k,i} \) are the eigenvalues of \( \hat{\rho}_i \). A d rank approximation of \( \hat{\rho}_i \) is therefore \( \hat{\rho}_{i,d} := \sum_{k=1}^{d} \lambda_{k,i} |\psi_{ki}\rangle \langle \psi_{ki}| \) and indeed

\[
\lim_{d \to \infty} \left\| \hat{\rho}_{i,d} - \hat{\rho}_i \right\|_1 = \lim_{d \to \infty} \left\| \sum_{k=d+1}^{\infty} \lambda_{k,i} |\psi_{ki}\rangle \langle \psi_{ki}| \right\|_1 \leq \lim_{d \to \infty} \sum_{k=d+1}^{\infty} |\lambda_{k,i}| = 0. \quad \quad (97)
\]

To proceed we must first demonstrate

\[
\min_{POVM} \sum_{i=1}^{N} \sum_{j \neq i}^{M} p_i Tr(\hat{M}_j \hat{\rho}_i \hat{M}_j^\dagger) = \lim_{d \to \infty} \min_{POVM} \sum_{i=1}^{N} \sum_{j \neq i}^{M} p_i Tr(\hat{M}_j \hat{\rho}_{i,d} \hat{M}_j^\dagger). \quad \quad (98)
\]

To show the above we need only show that

\[
\lim_{d \to \infty} \left\| \min_{POVM} \sum_{i=1}^{N} \sum_{j \neq i}^{M} p_i Tr(\hat{M}_j \hat{\rho}_i \hat{M}_j^\dagger) - \min_{POVM} \sum_{i=1}^{N} \sum_{j \neq i}^{M} p_i Tr(\hat{M}_j \hat{\rho}_{i,d} \hat{M}_j^\dagger) \right\| = 0. \quad \quad (99)
\]

We proceed as follows.

\[
\left\| \min_{POVM} \sum_{i=1}^{N} \sum_{j \neq i}^{M} p_i Tr(\hat{M}_j \hat{\rho}_i \hat{M}_j^\dagger) - \min_{POVM} \sum_{i=1}^{N} \sum_{j \neq i}^{M} p_i Tr(\hat{M}_j \hat{\rho}_{i,d} \hat{M}_j^\dagger) \right\| \leq \quad \quad (100)
\]

\[
\max_{POVM} \left\| \sum_{i=1}^{N} \sum_{j \neq i}^{M} p_i Tr(\hat{M}_j (\hat{\rho}_i - \hat{\rho}_{i,d}) \hat{M}_j^\dagger) \right\| \leq \quad \quad (101)
\]

\[
\max_{POVM} \sum_{i=1}^{N} \sum_{j \neq i}^{M} p_i \left\| \hat{M}_j (\hat{\rho}_i - \hat{\rho}_{i,d}) \hat{M}_j^\dagger \right\|_1 \leq \max_{POVM} \sum_{i=1}^{N} \sum_{j \neq i}^{M} p_i \left\| \hat{M}_j \right\|_\infty \| (\hat{\rho}_i - \hat{\rho}_{i,d}) \|_1 \left\| \hat{M}_j \right\|_\infty = \quad \quad (102)
\]
\[
\max_{POVM} \sum_{i=1}^{N} \sum_{j \neq i} p_i \left\| \hat{\rho}_i - \hat{\rho}_{i,d} \right\|_1 = \sum_{i=1}^{N} \sum_{j \neq i} p_i \left\| \hat{\rho}_i - \hat{\rho}_{i,d} \right\|_1 = \tag{103}\]

\[
\sum_{i=1}^{N} \sum_{j \neq i} p_i \sum_{k=d+1}^{\infty} |\lambda_{ki}| \leq N \sum_{i}^{\infty} p_i \sum_{k=d+1}^{\infty} |\lambda_{ki}| \tag{104}\]

and indeed
\[
\lim_{d \to \infty} N \sum_{i=1}^{N} p_i \sum_{k=d+1}^{\infty} \lambda_{ki} = \tag{105}\]
\[
N \sum_{i=1}^{N} p_i \lim_{d \to \infty} \sum_{k=d+1}^{\infty} \lambda_{ki} = N \sum_{j=1}^{N} 0 = 0. \tag{106}\]

Therefore,
\[
\lim_{d \to \infty} \left( \min_{POVM} \sum_{i=1}^{N} \sum_{j \neq i} p_i \text{Tr}\{\hat{\mathcal{M}}_j \hat{\rho}_i \hat{\mathcal{M}}_j^\dagger\} - \min_{POVM} \sum_{i,j \neq j} p_i \text{Tr}\{\hat{\mathcal{M}}_j \hat{\rho}_i \hat{\mathcal{M}}_j^\dagger\} \right) = 0 \tag{107}\]

which means that
\[
\min_{POVM} \sum_{i=1}^{N} \sum_{j \neq i} p_i \text{Tr}\{\hat{\mathcal{M}}_j \hat{\rho}_i \hat{\mathcal{M}}_j^\dagger\} = \lim_{d \to \infty} \min_{POVM} \sum_{i=1}^{N} \sum_{j \neq i} p_i \text{Tr}\{\hat{\mathcal{M}}_j \hat{\rho}_i \hat{\mathcal{M}}_j^\dagger\}. \tag{108}\]

Let us now introduce a normalization constant \(\alpha_{i,d} := \text{Tr}\{\hat{\rho}_{i,d}\}\). Using this normalization and the Knill-Barnum bound (30) \cite{11} we have
\[
\lim_{d \to \infty} \min_{POVM} \sum_{i,j \neq i} p_i \alpha_{i,d}^{-1} \text{Tr}\{\hat{\mathcal{M}}_j \alpha_{i,d} \hat{\rho}_{i,d} \hat{\mathcal{M}}_j^\dagger\} \leq \lim_{d \to \infty} \left( \min_{k} (\alpha_{k,d}^{-1}) \right) \min_{POVM} \sum_{i,j \neq i} p_i \text{Tr}\{\hat{\mathcal{M}}_j \alpha_{i,d} \hat{\rho}_{i,d} \hat{\mathcal{M}}_j^\dagger\} \leq \tag{109}\]
\[
\lim_{d \to \infty} \max_{k} (\alpha_{k,d}^{-1}) \sum_{i,j \neq i} \sqrt{p_i p_j} \sqrt{F(\alpha_{i,d} \hat{\rho}_{i,d}, \alpha_{j,d} \hat{\rho}_{j,d})} = \tag{110}\]
\[
\lim_{d \to \infty} \max_{k} (\alpha_{k,d}^{-1}) \sum_{i,j \neq i} \sqrt{p_i p_j} \left\| \sqrt{\alpha_{i,d} \hat{\rho}_{i,d}} \sqrt{\alpha_{j,d} \hat{\rho}_{j,d}} \right\|_1 = \tag{111}\]
\[
\sum_{i=1}^{N} \sum_{j \neq i} \sqrt{p_i p_j} \lim_{d \to \infty} \max_{k} (\alpha_{k,d}^{-1}) \sqrt{\alpha_{i,d} \alpha_{j,d}} \left\| \sqrt{\hat{\rho}_{i,d}} \sqrt{\hat{\rho}_{j,d}} \right\|_1 = \tag{112}\]
\[
\sum_{i=1}^{N} \sum_{j \neq i} \sqrt{p_i p_j} \left( \lim_{d \to \infty} \max_{k} (\alpha_{k,d}^{-1}) \sqrt{\alpha_{i,d} \alpha_{j,d}} \right) \left( \lim_{d \to \infty} \left\| \sqrt{\hat{\rho}_{i,d}} \sqrt{\hat{\rho}_{j,d}} \right\|_1 \right) = \tag{113}\]
\[
\sum_{i=1}^{N} \sum_{j \neq i} \sqrt{p_i p_j} \sqrt{F(\hat{\rho}_i, \hat{\rho}_j)} \tag{114}\]

where we have used lemma 3 and the fact that \(\lim_{d \to \infty} \alpha_{k,d} = 1\) for all \(k\) in the final equality.

\section{Quantum Chernoff Bounds From \cite{22}}

One of the main results in \cite{22} was the following relationship, useful for the study of Asymptotic QSD.
\[
\frac{1}{3} \xi_{QCB} \left( \{ \hat{\rho}_i \}_{i=1}^{N} \right) \leq - \lim_{n \to \infty} \frac{\log \left( \min_{POVM} p_E(n) \right)}{n} \leq \xi_{QCB} \left( \{ \hat{\rho}_i \}_{i=1}^{N} \right) \tag{115}\]
where $\xi_{QCB}$ is the quantum Chernoff bound for an $N$ mixture, defined as

$$
\xi_{QCB}\left(\{\hat{\rho}_i\}_{i=1}^N\right) := \min_{i,j} \xi_{QCB}(\hat{\rho}_i, \hat{\rho}_j) \tag{116}
$$

where

$$
\xi_{QCB}(\hat{\rho}_i, \hat{\rho}_j) = -\log \left( \min_{0 \leq s \leq 1} \text{Tr}\left\{ \hat{\rho}_s^i \hat{\rho}_j^{1-s} \right\} \right) \tag{117}
$$

References


[20] Scott M. Cohen Local approximation for perfect discrimination of quantum states (PHYSICAL REVIEW A 107, 012401 (2023))


