Open Quantum Systems

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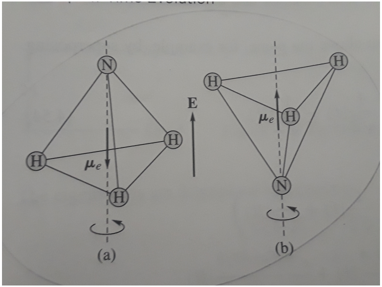
Introduction

- Quantum coherence
- Ammonia Molecule
- Spin Chains
- Spontaneous emission and the Born-Markov Master equation
- Collisional Decoherence
- DFS

The Ammonia Molecule



Hamiltonian

$$H \rightarrow \begin{pmatrix} E_0 & -\epsilon \\ -\epsilon & E_0 \end{pmatrix} = \begin{pmatrix} \langle 1|H|1\rangle & \langle 1|H|2\rangle \\ \langle 2|H|1\rangle & \langle 2|H|2\rangle \end{pmatrix}$$

Let us solve SE for $|\psi(0)\rangle = |1\rangle$

- $i\hbar\partial_t|\psi(t)\rangle = H|\psi(t)\rangle$
- $|\psi(t)\rangle = U(t)|\psi(0)\rangle$
- $U(t) = e^{\frac{-iHt}{\hbar}}$.

Ammonia Molecule continued

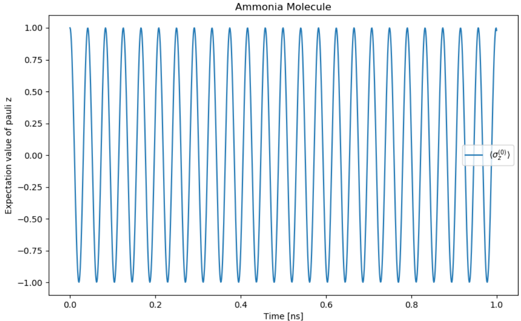
Solution

$$|\psi(t)\rangle = e^{\frac{-iE_0 t}{\hbar}}(\cos(\tfrac{\epsilon t}{\hbar})|1\rangle + i\sin(\tfrac{\epsilon t}{\hbar})|2\rangle)$$

State Matrix

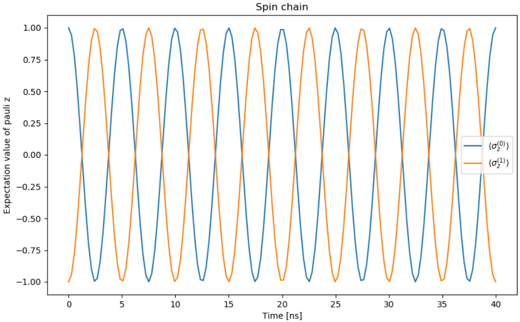
$$\rho(t) = |\psi(t)\rangle\langle\psi(t)| \rightarrow \begin{pmatrix} \cos^2(\frac{\epsilon t}{\hbar}) & -\cos(\frac{\epsilon t}{\hbar})\sin(\frac{\epsilon t}{\hbar}) \\ i\sin(\frac{\epsilon t}{\hbar})\cos(\epsilon t\hbar) & \sin^2(\frac{\epsilon t}{\hbar}) \end{pmatrix}$$

Time evolution of $Tr[\rho(t)\sigma_z]$



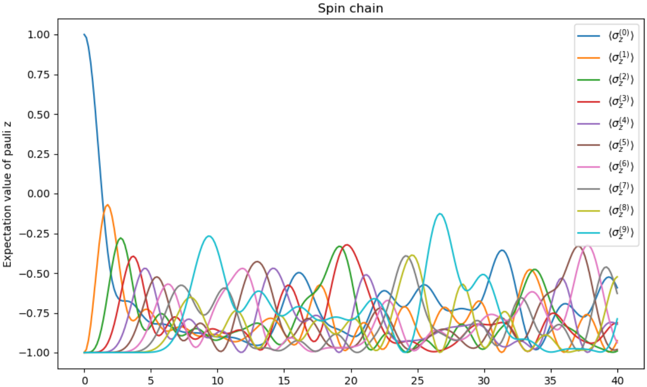
2 spin system

- $H = -\frac{1}{2}[h_0\sigma_z^0 + h_1\sigma_z^1] - \frac{1}{2}[J_x^0\sigma_x^0\sigma_x^1 + J_y^0\sigma_y^0\sigma_y^1 + J_z^0\sigma_z^0\sigma_z^1]$
- $H \in B(\mathbb{C}^{\otimes 4})$
- Let $N = 10$ and$|\psi(0)\rangle = |10\rangle \in \mathbb{C}^{\otimes 4}$



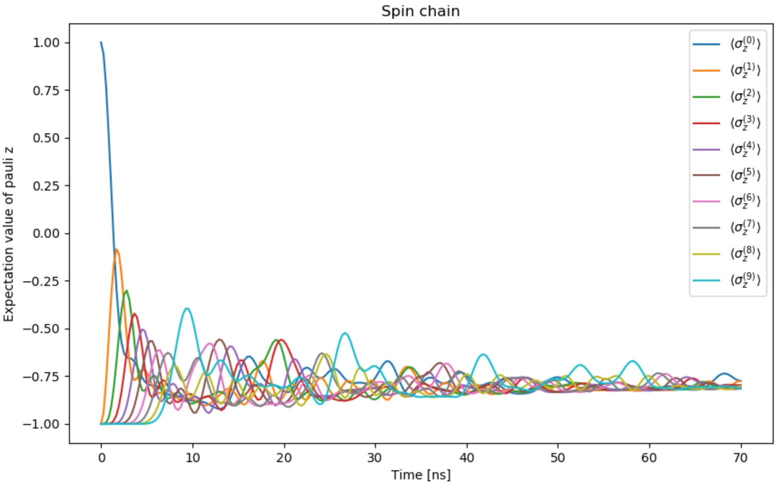
Spin chains

- $H = -\frac{1}{2}\sum_n^N h_n\sigma_z^n - \frac{1}{2}\sum_n^{N-1}[J_x^n\sigma_x^n\sigma_x^{n+1} + J_y^n\sigma_y^n\sigma_y^{n+1} + J_z^n\sigma_z^n\sigma_z^{n+1}]$
- $H \in B(\mathbb{C}^{\otimes 2N})$
- Solving SE here is hard. Let's use qutip!
- Let $N = 10$ and $|\psi(0)\rangle = |1000000000\rangle \in \mathbb{C}^{\otimes 2N}$



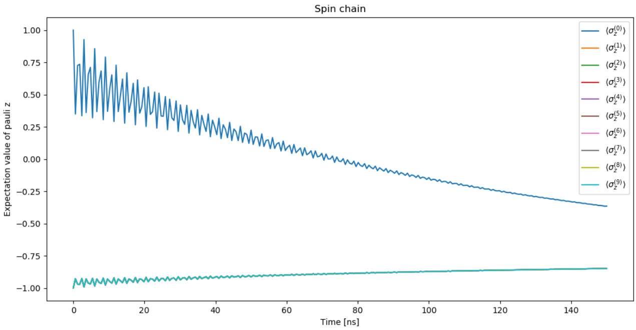
More on spin chains

Stronger coupling parameters



Central spin sysem

- $H = -\frac{1}{2}\sum_n^N h_n\sigma_z^n - \frac{1}{2}\sum_n^{N-1}[J_x^n\sigma_x^0\sigma_x^{n+1} + J_y^n\sigma_y^0\sigma_y^{n+1} + J_z^n\sigma_z^0\sigma_z^{n+1}]$
- $H \in B(\mathbb{C}^{\otimes 2N})$
- Again, solving SE here is hard. Let's use qutip!
- Let $N = 10$ and $|\psi(0)\rangle = |1000000000\rangle \in \mathbb{C}^{\otimes 2N}$



Quantum open systems

- Total system has some *Hilbert* space $\mathscr{H}_S \otimes \mathscr{H}_E$
- $|\psi_{SE}\rangle \in \mathscr{H}_S \otimes \mathscr{H}_E$
- Dynamics provided by Schrödinger's equation. $i\hbar\partial_t|\psi_{SE}(t)\rangle = H|\psi_{SE}(t)\rangle$ where $H = H_S + H_E + H_I$.
- $|\psi_{SE}(t)\rangle = e^{-\frac{it}{\hbar}H}$. Just like before. We can attain the reduced

  dynamics by partial tracing over the degrees of freedom pertaining to the environment. i.e.

$$Tr_E\{|\psi_{SE}(t)\rangle\langle\psi_{SE}(t)|$$

Partial Trace

Definition

$Tr_E\{\} : T(\mathscr{H}_S \otimes \mathscr{H}_E) \to T(\mathscr{H}_S)$

$$Tr_E\{|\psi_{SE}(t)\rangle\langle\psi_{SE}(t)|\} := \sum_k \langle\phi_k|\psi_{SE}(t)\rangle\langle\psi_{SE}(t)|\phi_k\rangle,$$

where $\{|\phi_k\rangle\}_k$ is an ONB for $\mathscr{H}_E$.

- Let use make sure that this map is the correct one.
- $$A_S \to A_S \otimes I_E,$$

  $$\langle A_S \otimes I_E\rangle = Tr\{\rho_{SE}(A_S \otimes I_E)\}.$$

  But it can be shown that

  $$Tr\{\rho_{SE}(A_S \otimes I_E)\} = Tr_S\{\rho_S A_S\}!!!!.$$

Partial trace

- There you have it, all we need to do is compute

$$Tr_E\{|\psi_{SE}(t)\rangle\langle\psi_{SE}(t)|.$$

- This is generally intractable and approximations must be made.

- Let's switch to the equivalent approach, Von neumann equation is starting point.

$$\frac{\partial}{\partial t}\rho_{SE}(t) = -\frac{i}{\hbar}[H_{SE}, \rho_{SE}(t)] \tag{1}$$

We can attain the reduced dynamics by partial trace.

$$\frac{\partial}{\partial t}\rho_S(t) = -\frac{i}{\hbar}Tr_E\{[H_{SE}, \rho_{SE}(t)]\} \tag{2}$$

Born-Markov approximation

Equivalently.

$$\frac{\partial}{\partial t}\rho_S(t) = -\frac{i}{\hbar} Tr_E\{[H_{SE}, \rho_S(0) \otimes \rho_E(0)]\} +$$

$$+\frac{i^2}{\hbar^2} \int_0^t dt_1 \, Tr_E\{[H_{SE}, [H_{SE}, \rho_S(t_1) \otimes \rho_E(0)]]\}$$

- Here we already begin to make approximations. We have assumed that $\rho_{SE}(0) = \rho_S(0) \otimes \rho_E(0)$, separable. This is the *Born* approximation.
- We have also assumed that the environment is unchanged by the system S. Unfortunately this too is intractable if we are

  not able to make the approximation $\rho_S(t_1) \to \rho_S(t)$. This makes the integro differential equation above more manageable.

two-level system in bath

- Let E be a large, with respect to the system, bosonic bath and S be a two-level system. Assuming that the two level system is in the excited state at $t = 0$ we can use the Born-Markov approximation to arrive at the following equation.
- $$\frac{\partial}{\partial t}\rho_S(t) = \frac{-i}{2}(\omega_a + \Delta\omega_a)[\sigma_z, \rho_S(t)] + \gamma D[\sigma_-]\rho_S(t). \quad (3)$$
- $D[\sigma_-]\rho = \sigma_-\rho\sigma_+ - \frac{1}{2}(\sigma_+\sigma_-\rho + \rho\sigma_+\sigma_-)$ This is know as the $Born - Markov$ master equation. We now use it to solve for the reduced density matrix.
- First assume that $\rho_S(t) = \frac{1}{2}[I_2 + x(t)\sigma_x + y(t)\sigma_y + z(t)\sigma_z]$, we also constraint the scalar functions to $Tr_S\{\rho_S(t)\} = 1$.

- $\frac{\partial}{\partial t} z(t) = Tr\{\sigma_z \frac{\partial}{\partial t} \rho_S(t)\}$
- $\frac{\partial}{\partial t} y(t) = Tr\{\sigma_y \frac{\partial}{\partial t} \rho_S(t)\}$
- $\frac{\partial}{\partial t} x(t) = Tr\{\sigma_x \frac{\partial}{\partial t} \rho_S(t)\}$

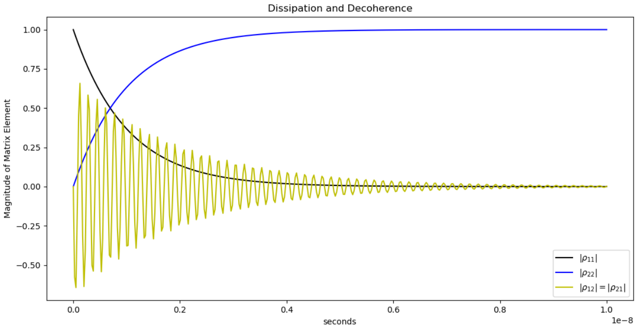
Using the *Lindblad* Master equation to substitute for $\frac{\partial}{\partial t} \rho_S(t)$ these equations become

- $\frac{\partial}{\partial t} z(t) = -\gamma(z(t) + 1)$
- $\frac{\partial}{\partial t} y(t) = (\Delta\omega_a)x(t) - \frac{\gamma}{2} y(t)$
- $\frac{\partial}{\partial t} x(t) = -(\Delta\omega_a)y(t) - \frac{\gamma}{2} x(t)$

with solutions

- $z(t) = 2e^{-\gamma t} - 1$
- $y(t) = -e^{-\frac{\gamma t}{2}} \sin((\omega_a + \Delta\omega_a)t)$
- $x(t) = e^{-\frac{\gamma t}{2}} \sin((\omega_a + \Delta\omega_a)t).$

Continued



$$\rho_S(t) \to \begin{bmatrix} e^{-\gamma t} & e^{-\frac{\gamma t}{2}} \sin((\omega_a + \Delta_a)t)\frac{(1+i)}{2} \\ e^{-\frac{\gamma t}{2}} \sin((\omega_a + \Delta_a)t)\frac{(1-i)}{2} & 1 - e^{-\gamma t} \end{bmatrix}$$

Collisional Decoherence

Set up

Let $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$ be a product *Hilbert* space.

- $dim[\mathscr{H}_A] = N$
- $dim[\mathscr{H}_B] = M$
- $dim[\mathscr{H}] = NM$.

Take $H \in \mathscr{B}(\mathscr{H}) = \mathscr{B}(\mathscr{H}_A \otimes \mathscr{H}_B)$ with the following form.

$$H = \sum_k S_k \otimes E_k \tag{4}$$

- $S_k \in \mathscr{B}(\mathscr{H}_A)$
- $E_K \in \mathscr{B}(\mathscr{H}_B)$
- $S_k = S_k^\dagger$ and $E_k = E_k^\dagger$, self adjoint.

Recognize the following equation? $i\frac{\partial}{\partial t}\psi_t = H\psi_t$, $\psi_t \in \mathscr{H}$.

Set Up part 2

The solution to SE in the time independent case is just

$$\psi_t = e^{\frac{-it}{\hbar}\sum_k S_k \otimes E_k}\psi_0 \tag{5}$$

Assuming that $\psi_0 = [\sum_l c_l \phi_l] \otimes \eta_0$,

- $\{\phi_i\}_i$ form an ONB for $\mathscr{H}_A$.
- $\eta_0 \in \mathscr{H}_B$

$$\psi_t = e^{\frac{-it}{\hbar}\sum_k S_k \otimes E_j}([\sum_l c_l \phi_l] \otimes \eta_0) \tag{6}$$

Let us move to the state matrix representation.

- $\rho_t = \psi_t \psi_t^\dagger =$
  $e^{\frac{-it}{\hbar}\sum_k S_k \otimes E_k}([\sum_{l,m} c_l c_m^* \phi_l \phi_m^\dagger] \otimes \eta_0 \eta_0^\dagger) e^{\frac{it}{\hbar}\sum_k S_k \otimes E_k}$

Decoherence

Partial tracing over the degrees of freedom pertaining to $\mathscr{H}_B$ we get the $\mathscr{H}_A$ local non unitary equation.

$$\rho_S(t) = Tr_B[e^{\frac{-it}{\hbar}\sum_k S_k \otimes E_k}([\sum_{l,m} c_l c_m^* \phi_l \phi_m^\dagger] \otimes \eta_0 \eta_0^\dagger) e^{\frac{it}{\hbar}\sum_k S_k \otimes E_k}] \quad (7)$$

This partial trace in general reduces to some state of the form,

$$\rho_S(t) = \sum_{l,m} a_l(t) a_m^*(t) \phi_l \phi_m^\dagger$$

with $a_l(t)a_m^*(t) \to 0$ as $t \to \infty$ for$l \neq m$.

Question

From what space $\mathscr{H}_C \subset \mathscr{H}_A$ may we construct sperpositions $\sum_l c_l\phi_l$ that are immune to decoherence? i.e.

$$Tr_B[e^{\frac{-it}{\hbar}\sum_k S_k\otimes E_k}([\sum_{l,m} c_l c_m^* \phi_l\phi_m^\dagger]\otimes\eta_0\eta_0^\dagger)e^{\frac{it}{\hbar}\sum_k S_k\otimes E_k}] = \sum_{l,m} c_l c_m^* \phi_l\phi_m^\dagger \tag{8}$$

Need$\{\phi_i\}_i$ ONB, with the exotic property of forming a degenerate

eigen space for all S_k.

$$\psi_t = e^{\frac{-it}{\hbar}\sum_k S_k\otimes E_k}[\sum_l c_l\phi_l] \otimes \eta_0 = \tag{9}$$

$$= [\sum_l c_l e^{\frac{-it}{\hbar}\sum_k \lambda_k I_A\otimes E_k}\phi_l] \otimes \eta_0 = \sum_l c_l\phi_l \otimes [e^{\frac{-it}{\hbar}\sum_k \lambda_k I_A\otimes E_k}\eta_0] \tag{10}$$

Partial Trace

Let us now partial trace the corresponding density matrix.

$$\rho_S(t) = \sum_{l,m} c_l c_m^* \phi_l \phi_m^\dagger Tr_B[e^{\frac{-it}{\hbar}\sum_k \lambda_k I_A \otimes E_k} \eta_0 \eta_0^\dagger e^{\frac{it}{\hbar}\sum_k \lambda_k I_A \otimes E_k}]$$

The trace term is just one since density matricese have trace one under unitary evolution.

$$\rho_S(t) = \sum_{l,m} c_l c_m^*$$

. :)

Example, symmetric dephasing

Consider a system of N qubits coupled to its environment in the follwoing way.

$$|0\rangle_j \rightarrow |0\rangle_j \tag{11}$$

$$|1\rangle_j \rightarrow e^{i\phi}|1\rangle_j. \tag{12}$$

j indexes over all qubits. Let the initial state be

$$|\psi\rangle_0 = \bigotimes_{j=1}^{N}(a_j|0\rangle_j + b_j|1\rangle_j).$$

The dephasing process evolves our system into the following state.

$$|\psi\rangle_\phi = \bigotimes_{j=1}^{N}(a_j|0\rangle_j + b_j e^{i\phi}|1\rangle_j)$$

with a probability p_ϕ

Example continued

The ensemble $\{|\psi\rangle_\phi, p_\phi\}$ can be expressed equivalently as a mixed state.

$$\rho = \int p_\phi |\psi\rangle_\phi \langle\psi| d\phi$$

$$|\psi\rangle_\phi\langle\psi| \rightarrow \left[\begin{array}{cc} |a_j|^2 & a_j b_j^* e^{-i\phi} \\ a_j^* b_j e^{i\phi} & |b|^2 \end{array} \right]. \tag{13}$$

For a gaussian distribution $p_\phi = (4\pi\alpha^{\frac{-1}{2}})e^{\frac{phi^2}{4\alpha}}$ we have

$$\left[\begin{array}{cc} |a_j|^2 & a_j b_j^* e^{-\alpha} \\ a_j^* b_j e^{-\alpha} & |b|^2 \end{array} \right]. \tag{14}$$

There is indeed decoherence present, lets look for a DFS.

Example continued

For starters lets consider the case $N = 2$. The dephasing for each of the constituents of the corresponding *Hilbert* space $\mathbb{C}^2 \otimes \mathbb{C}^2$ is summarized by the following.

$$|00\rangle \to |00\rangle \tag{15}$$

$$|01\rangle \to e^{i\phi}|01\rangle \tag{16}$$

$$|10\rangle \to e^{i\phi}|10\rangle \tag{17}$$

$$|11\rangle \to e^{2i\phi}|11\rangle. \tag{18}$$

$$Span\{|01\rangle, |10\rangle\}?$$

check...

$$|\psi\rangle = a|01\rangle + b|10\rangle \to ae^{i\phi}|01\rangle + be^{i\phi}|10\rangle = e^{i\phi}|\psi\rangle$$

It works!!

Example continued

For $N = 3$ the largest DFS is $Span\{|001\rangle, |010\rangle, |100\rangle\}$ or

$Span\{|011\rangle, |101\rangle, |110\rangle\}$

In general $max[dim(DFS)] = \binom{N}{F(\frac{N}{2})}$ A textbook application of

stirling's formula yields the following.

$$\frac{|max[Dim(DFS)] - 2^N|}{2^N} \to 1.$$

The dimension of the optimal DFS becomes relatively close to the dimenstion of the system for large N.