Spectrum Broad Cast Structures from von Neumann type interaction Hamiltonians

Alberto Acevedo[1], Jarek Korbicz[2], Janek Wehr [1]

[1]**The University of Arizona, Tucson, USA**

[2]**Center for Theoretical Physics, Polish Academy of Science, Warsaw, Poland**

Abstract: In this paper we develop the mathematical foundations of the recently established theory of Spectrum Broadcast Structures (SBS); we do this for the case of a central system interacting with N non-interacting environments (von Neumann-type interactions). We developed tools for studying the dynamical convergence of the multipartite quantum states in question to an SBS state and provide necessary conditions for SBS to arise dynamically from the ensuing non-unitary dynamics.

1 Work by Jarek et all

In recent times significant attention has been given to a family of multipartite states named *Spectrum Broadcast Structures* (SBS) [37] [38] [39] [52]. Since its genesis, the theory of SBS has been used as a tool in the discipline of *Quantum Foundations*; particularly in the theories of *Quantum Decoherence* and *Quantum Darwinism*[12][30][40][41]. Recently, quantum darwinism and SBS theory have been shown to be equivalent under certain technical assumptions [42]. Motivating the theory of quantum darwinism and the theory of SBS is the question of objectivity in the quantum world. To avoid philosophical contention [37] [38] and[39] provide a definition of objectivity motivated by properties of classical dynamical systems. A multipartite quantum mechanical state satisfying such properties is called a SBS. The definition of objectivity proposed in [37] is:

Definition 1. *A state of the system S exists objectively if many observers can find out the state of S independently, and without perturbing it.*

There are two clauses in the definition above that are ambiguous, namely, "can find out the state of S" and "without perturbing it". The first of these means that any of the observers may locally solve a *Quantum State Discrimination* optimization problem (QSD) [22] [25] [63] that allows the observer to identify the state of the system S by proxy, we include a brief discussion regarding QSD in appendix A. The second clause, "without perturbing it" may be formalized by introducing a distance measure. We will only be using the trace distance, but different distance measures may be more relevant in other scenarios. The following definition proposed in [37] is a mathematical formalization of Definition 1 and is what we will refer to as a SBS.

Definition 2. *SBS: A Spectrum Broadcast Structure is a multipartite state (also called joint state) of a central system S and an environment E, consisting of sub-environments $E^1, E^2, ..., E^{N_E}$:*

$$\hat{\boldsymbol{\rho}} = \sum_i p_i |i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_i^{E^k} \tag{1}$$

where $\{|i\rangle\}_i$ is some basis in the system's space, p_i are probabilities summing to one, and all states

$\hat{\boldsymbol{\rho}}_i^{E^k}$ *are perfectly distinguishable in the following sense:*

$$F\big(\hat{\boldsymbol{\rho}}_i^{E^k}, \hat{\boldsymbol{\rho}}_j^{E^k}\big) = 0 \tag{2}$$

for all $i \neq j$ *and for all* $k = 1, ..., N_E$*. Where* $F(..., ...)$ *is the quantum fidelity defined as* $F\big(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\sigma}}\big) := \big\|\sqrt{\hat{\boldsymbol{\rho}}}\sqrt{\hat{\boldsymbol{\sigma}}}\big\|_1^2$ *[60].*

In [37] it is argued that SBS satisfies the desired definition of objectivity and that it is the only structure that satisfies such a definition. The argument for why the observer monitoring (i.e. employing measurements characterized via a *POVM* [9]) E^l may find out the state of S independently is the following. Let us analyze the local state pertaining to E^l; to do this we partially trace out the degrees of freedom pertaining to the system S and all of the environments E^k with the exception of the lth environment. i.e. from (1) we obtain

$$\sum_i p_i \left(\prod_{k \neq l} Tr_{E^k}\{\hat{\boldsymbol{\rho}}_i^{E^k}\} \right) \langle i|i\rangle \hat{\boldsymbol{\rho}}_i^{E^l} = \sum_i p_i \hat{\boldsymbol{\rho}}_i^{E^l}. \tag{3}$$

Notice that this is a mixed state. If $F\big(\hat{\boldsymbol{\rho}}_i^{E^l}, \hat{\boldsymbol{\rho}}_j^{E^l}\big) = 0$ for all $i \neq j$ then the associated quantum state discrimination problem may be fully solved. This means that there exists a POVM which the observer monitoring the environment E^l may utilize to conduct measurements on E^l yielding perfect distinguishability between the possible outcomes of the mixture (3). Furthermore, the state $\hat{\boldsymbol{\rho}}_i^{E^l}$ is correlated with the state $\big|i\big\rangle\big\langle i\big|$ of S in the sense that when S is found to be in the state $\big|i\big\rangle\big\langle i\big|$ the lth environment will be found in the state $\hat{\boldsymbol{\rho}}_i^{E^l}$. Owing to the perfect distinguishability between the states $\hat{\boldsymbol{\rho}}_i^{E^l}$ for all i, there is no ambiguity regarding the state of S given that E^l is found to be in the state $\hat{\boldsymbol{\rho}}_i^{E^l}$. Since l was taken to be arbitrary, it is clear that any environmental observer may find out the state of S faithfully so long as $F\big(\hat{\boldsymbol{\rho}}_i^{E^l}, \hat{\boldsymbol{\rho}}_j^{E^l}\big) = 0$ is satisfied for all $i \neq j$.

To argue non-disturbance (a similar approach follows for approximate non-disturbance) we first re-emphasize that the "can find out" in Definition 1 formally means that for every E^k there exists a POVM $\{\hat{\mathbf{E}}_i^{E^k}\}_i$ that solves the respective local QSD problem, i.e. that discriminates perfectly the mixture (3). $\{\bigotimes_{k=1}^{N_E} \hat{\mathbf{E}}_{i_k}^{E^k}\}_{i_1, i_2, ..., i_{N_E}}$ will hence be a POVM acting on $\mathcal{S}\big(\mathscr{H}_S \bigotimes_{k=1}^{N_E} \mathscr{H}_{E^k}\big)$. If the POVM optimally solving the local QSD problem for each environment E^l does so in a non-perturbing way, i.e. not changing the state after the associated measurement quantum channel has been applied in the trace distance sense, then the measurement associated with the POVM $\{\bigotimes_{k=1}^{N_E} \hat{\mathbf{E}}_{i_k}^{E^k}\}_{i_1, i_2, ..., i_{N_E}}$ may be shown to also be non-disturbing with respect to the trace distance. i.e. it can be shown that

$$\frac{1}{2}\left\| \sum_i p_i |i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_i^{E^k} - \sum_{i_1}\sum_{i_2}...\sum_{i_{N_E}} \left(\sum_i p_i |i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{M}}_{i_k}^{E^k} \hat{\boldsymbol{\rho}}_i^{E^k} \left(\hat{\mathbf{M}}_{i_k}^{E^k}\right)^\dagger \right) \right\|_1 = 0 \tag{4}$$

where $\left(\hat{\mathbf{M}}_{i_k}^{E^k}\right)^\dagger \hat{\mathbf{M}}_{i_k}^{E^k} = \hat{\mathbf{E}}_{i_k}^{E^k}$ isthe POVM perfectly discriminating the mixture $\sum_i p_i \hat{\boldsymbol{\rho}}_i^{E^k}$. To show (4) note that perfect distinguishability of the $\hat{\boldsymbol{\rho}}_i^{E^k}$ for all k implies that we may devise a POVM $\{\hat{\mathbf{E}}_{i_k}^{E^k}\}_{i_k}$ such that

$$\hat{\mathbf{M}}_{i_k}^{E^k} \hat{\boldsymbol{\rho}}_i^{E^k} \left(\hat{\mathbf{M}}_{i_k}^{E^k}\right)^\dagger = \delta_{i_k i}. \tag{5}$$

owing to the non-overlapping support of the $\hat{\boldsymbol{\rho}}_i^{E^k}$. With (5) in mind, we may estimate the trace distance in (4).

$$\frac{1}{2}\Big\| \sum_i p_i|i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_i^{E^k} - \sum_{i_1}\sum_{i_2}\dots\sum_{i_{N_E}}\Big(\sum_i p_i|i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{M}}_{i_k}^{E^k}\hat{\boldsymbol{\rho}}_i^{E^k}\left(\hat{\mathbf{M}}_{i_k}^{E^k}\right)^\dagger\Big)\Big\|_1 = \tag{6}$$

$$\frac{1}{2}\Big\| \sum_i p_i|i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_i^{E^k} - \sum_i p_i|i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{M}}_i^{E^k}\hat{\boldsymbol{\rho}}_i^{E^k}\left(\hat{\mathbf{M}}_i^{E^k}\right)^\dagger\Big\|_1 \leq \tag{7}$$

$$\frac{1}{2}\sum_i p_i\Big\| \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_i^{E^k} - \bigotimes_{k=1}^{N_E} \hat{\mathbf{M}}_i^{E^k}\hat{\boldsymbol{\rho}}_i^{E^k}\left(\hat{\mathbf{M}}_i^{E^k}\right)^\dagger\Big\|_1 \tag{8}$$

To proceed we introduce the following Lemma and we also take this opportunity to introduce two results that shall be used in the following.

Lemma 1. *Telescopic inequality [39]: Let $\hat{\mathbf{A}}^k$ and $\hat{\hat{B}}^k$ be trace class operators for all k. Then,*

$$\|\bigotimes_{k=1}^{N} \hat{\mathbf{A}}^k - \bigotimes_{k=1}^{N} \hat{\mathbf{B}}^k\|_1 \leq \tag{9}$$

$$\sum_{j=1}^{N}\left(\prod_{k=1}^{j-1}\|\hat{\mathbf{A}}^k\|_1\right) \times \|\hat{\mathbf{A}}^j - \hat{\mathbf{B}}^j\|_1 \times \left(\prod_{k=j+1}^{N}\|\hat{\mathbf{B}}^k\|_1\right) \tag{10}$$

Theorem 1. *Montanaro Bound [68]:*

$$\min_{POVM} p_E \geq \frac{1}{2}\sum_i\sum_{j;j\neq i} p_i p_j F(\hat{\boldsymbol{\rho}}_i, \hat{\boldsymbol{\rho}}_j) \tag{11}$$

Theorem 2. *Knill and Barnum [26]*

$$\min_{POVM} p_E \leq \sum_i\sum_{j;j\neq i} \sqrt{p_i p_j}\sqrt{F(\hat{\boldsymbol{\rho}}_i, \hat{\boldsymbol{\rho}}_j)} \tag{12}$$

Using Lemma 1, (8) may be bounded as follows.

$$\frac{1}{2}\sum_i p_i\Big\| \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_i^{E^k} - \bigotimes_{k=1}^{N_E} \hat{\mathbf{M}}_i^{E^k}\hat{\boldsymbol{\rho}}_i^{E^k}\left(\hat{\mathbf{M}}_i^{E^k}\right)^\dagger\Big\|_1 \leq \frac{1}{2}\sum_{k=1}^{N_E}\sum_i p_i\Big\|\hat{\boldsymbol{\rho}}_i^{E^k} - \hat{\mathbf{M}}_i^{E^k}\hat{\boldsymbol{\rho}}_i^{E^k}\left(\hat{\mathbf{M}}_i^{E^k}\right)^\dagger\Big\|_1 \tag{13}$$

We claim that the distinguishability criterion $F(\hat{\boldsymbol{\rho}}_i^{E^k}, \hat{\boldsymbol{\rho}}_j^{E^k}) = 0$ ($i \neq j$) for all k is a necessary and sufficient condition for (13) to vanish. For the case of perfect distinguishability, the sufficiency is immediately clear since each $\hat{\mathbf{E}}_i^{E^k}$ may be chosen to be a projector onto the domain of $\hat{\boldsymbol{\rho}}_i^{E^k}$ respectively, meaning that $\hat{\mathbf{M}}_i^{E^k}\hat{\boldsymbol{\rho}}_i^{E^k}\left(\hat{\mathbf{M}}_i^{E^k}\right)^\dagger = \hat{\boldsymbol{\rho}}_i^{E^k}$ which in turn implies that $\|\hat{\boldsymbol{\rho}}_i^{E^k} - \hat{\mathbf{M}}_i^{E^k}\hat{\boldsymbol{\rho}}_i^{E^k}\left(\hat{\mathbf{M}}_i^{E^k}\right)^\dagger\|_1 = 0$. The argument becomes more transparent in the case where all of the $\hat{\boldsymbol{\rho}}_i^{E^k}$ are projectors. In such a case we simply choose $\hat{\mathbf{E}}_i^{E^k} = \hat{\boldsymbol{\rho}}_i^{E^k}$.

The distinguishability condition $F(\hat{\boldsymbol{\rho}}_i^{E^k}, \hat{\boldsymbol{\rho}}_j^{E^k}) = 0$ $(i \neq j)$ for all k is of course an idealization; in practice there will always be some error involved in the distinguishability measures $F(\hat{\boldsymbol{\rho}}_i^{E^k}, \hat{\boldsymbol{\rho}}_j^{E^k}) = \varepsilon_k$ for all k, where $\varepsilon_k > 0$ will depend on dynamical parameters such as time. In such a case we must tread more carefully. In the previous paragraph, we did not need to calculate or estimate the trace norm present because we showed that the operator in the trace norm was the zero operator. If the perfect distinguishability condition is not satisfied, then we will need to compute/estimate the sum over i of trace norms in (13). Although it is known that

$$\min_{POVM} Tr\{\hat{\boldsymbol{\rho}}_i^{E^k} - \hat{\mathbf{M}}_i^{E^k}\hat{\boldsymbol{\rho}}_i^{E^k}\left(\hat{\mathbf{M}}_i^{E^k}\right)^\dagger\} \leq \sum_i \sum_{j;j\neq i}\sqrt{p_i p_j}\sqrt{F(\hat{\boldsymbol{\rho}}_i, \hat{\boldsymbol{\rho}}_i)} \tag{14}$$

via an application of 2 for all k, this does not aid us in the estimation of the associate optimization problem $\sum_i p_i \|\hat{\boldsymbol{\rho}}_i^{E^k} - \hat{\mathbf{M}}_i^{E^k}\hat{\boldsymbol{\rho}}_i^{E^k}\left(\hat{\mathbf{M}}_i^{E^k}\right)^\dagger\|_1$. It is clear by (14) that if the fidelities $F(\hat{\boldsymbol{\rho}}_i^{E^k}, \hat{\boldsymbol{\rho}}_j^{E^k})$ $(i \neq j)$ are arbitrarily small, then for all k the local QSD error $\min_{POVM} Tr\{\hat{\boldsymbol{\rho}}_i^{E^k} - \hat{\mathbf{M}}_i^{E^k}\hat{\boldsymbol{\rho}}_i^{E^k}\left(\hat{\mathbf{M}}_i^{E^k}\right)^\dagger\}$ will also become arbitrarily small. To show that a similar argument holds for the right-hand side of the inequality (13) we will prove in the following section a bound for (8) that will depend only on fidelities between the set of density operators $\{\hat{\boldsymbol{\rho}}_i^{E^k}\}_i$, and vanish as the fidelities $F(\hat{\boldsymbol{\rho}}_i^{E^k}, \hat{\boldsymbol{\rho}}_j^{E^k})$ $(i \neq j)$ decay to zero for all k. We will first show this for the case where the $\hat{\boldsymbol{\rho}}_i^{E^k}$ are pure states and then make strides in generalizing our results to the case where these operators are mixtures.

2 Bounding the Super Quantum State Discrimination Problem (SQSD)

For this section, we shall be simplifying our notational conventions since we shall not need the superscripts on the density operators used in the previous section. Consider the mixed state $\sum_{i=1}^N p_i \hat{\boldsymbol{\rho}}_i$, where $\sum_{i=1}^N p_i = 1$ and the $\hat{\boldsymbol{\rho}}_i$ are pure states in a Hilbert space of dimension greater than N, i.e. one-dimensional projections $|\psi_i\rangle\langle\psi_i|$, where $\{|\psi_i\rangle\}_{i=1}^N$ are normalized vectors. Assuming that $|\psi_i\rangle$ are linearly independent, we may use the well-known Gram-Schmidt procedure to define an associated orthonormal set.

Definition 3. *Gram-Schmidt Procedure: Assume that the set $\{|\psi\rangle_i\}_{i=1}^N$, of vectors in some vector space V, is a linearly independent set. Then the following construction yields an orthonormal set.*

$$|\phi_1\rangle = |\psi_1\rangle \tag{15}$$

$$|\phi_2\rangle = \frac{1}{\alpha_2}\Big(|\psi_2\rangle - \langle\phi_1|\psi_2\rangle|\phi_1\rangle\Big) \tag{16}$$

$$\vdots$$

$$|\phi_N\rangle = \frac{1}{\alpha_N}\Big(|\psi_N\rangle - \sum_{k=1}^{N-1}\langle\phi_k|\psi_N\rangle|\phi_k\rangle\Big) \tag{17}$$

Here $\alpha_i := \Big\||\psi_i\rangle - \sum_{k=1}^{i-1}\Big\langle\phi_k|\psi_i\rangle|\phi_k\Big\rangle\Big\| = \sqrt{1 - \sum_{k=1}^{i-1}|\langle\phi_k|\psi_i\rangle|^2}$ for $i > 1$ and $\alpha_1 = 1$ are the respective normalization constants. We have $Span\Big\{\{|\psi_i\rangle\}_{i=1}^N\Big\} = Span\Big\{\{|\phi_i\rangle\}_{i=1}^N\Big\}$.

The orthonormal set $\{|\phi_i\rangle\}_{i=1}^N$ may be used for the construction of a PVM, namely

$$\Big\{|\phi_i\rangle\langle\phi_i|\Big\}_{i=1}^N \cup \Big\{\mathbb{I} - \sum_{i=1}^N |\phi_i\rangle\langle\phi_i|\Big\} \tag{18}$$

which we will use it to estimate $\min_{POVM}\sum_{i=1}^{N}p_i\big\|\hat{\boldsymbol{\rho}}_i-\hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{M}}_i^\dagger\big\|_1$, this minimization problem will be named the *Super Quantum State Discrimination* problem (SQSD) due to its bounding of the associated QSD problem (i.e. $Tr\{\hat{\mathbf{A}}\}\leq\|\hat{\mathbf{A}}\|_1$):

$$\min_{POVM}\sum_{i=1}^{N}p_iTr\Big\{\hat{\boldsymbol{\rho}}_i-\hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{M}}_i^\dagger\Big\}\leq\min_{POVM}\sum_{i=1}^{N}p_i\big\|\hat{\boldsymbol{\rho}}_i-\hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{M}}_i^\dagger\big\|_1\leq \tag{19}$$

$$\min_{PVM}\sum_{i=1}^{N}p_i\big\|\hat{\boldsymbol{\rho}}_i-\hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{P}}_i\big\|_1\leq\sum_{i=1}^{N}p_i\Big\|\hat{\boldsymbol{\rho}}_i-|\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i|\phi_i\rangle\langle\phi_i|\Big\|_1 \tag{20}$$

Lemma 2. *Let $\hat{\boldsymbol{\rho}}_i$ and $|\phi_i\rangle$ be defined as above; also let $i>1$, then*

$$\big\|\hat{\boldsymbol{\rho}}_i-|\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i|\phi_i\rangle\langle\phi_i|\big\|_1\leq 2\sum_{k=1}^{i-1}|\langle\phi_k|\psi_i\rangle| \tag{21}$$

Proof.

$$\Big\|\hat{\boldsymbol{\rho}}_i-|\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i|\phi_i\rangle\langle\phi_i|\Big\|_1=\Big\||\psi_i\rangle\langle\psi_i|\hat{\boldsymbol{\rho}}_i|\psi_i\rangle\langle\psi_i|-|\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i|\phi_i\rangle\langle\phi_i|\Big\|_1= \tag{22}$$

$$\Big\|\Big(|\psi_i\rangle\langle\psi_i|-|\phi_i\rangle\langle\phi_i|\Big)\hat{\boldsymbol{\rho}}_i|\psi_i\rangle\langle\psi_i|+|\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i\Big(|\psi_i\rangle\langle\psi_i|-|\phi_i\rangle\langle\phi_i|\Big)\Big\|_1\leq \tag{23}$$

$$\Big\|\Big(|\psi_i\rangle\langle\psi_i|-|\phi_i\rangle\langle\phi_i|\Big)\hat{\boldsymbol{\rho}}_i|\psi_i\rangle\langle\psi_i|\Big\|_1+\Big\||\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i\Big(|\psi_i\rangle\langle\psi_i|-|\phi_i\rangle\langle\phi_i|\Big)\Big\|_1\leq \tag{24}$$

$$\Big\||\psi_i\rangle\langle\psi_i|-|\phi_i\rangle\langle\phi_i|\Big\|_1\Big\|\hat{\boldsymbol{\rho}}_i|\psi_i\rangle\langle\psi_i|\Big\|_1+\Big\||\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i\Big\|_1\Big\||\psi_i\rangle\langle\psi_i|-|\phi_i\rangle\langle\phi_i|\Big\|_1= \tag{25}$$

$$\Big\||\psi_i\rangle\langle\psi_i|-|\phi_i\rangle\langle\phi_i|\Big\|_1\left(\Big\|\hat{\boldsymbol{\rho}}_i|\psi_i\rangle\langle\psi_i|\Big\|_1+\Big\||\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i\Big\|_1\right)\leq \tag{26}$$

$$\Big\||\psi_i\rangle\langle\psi_i|-|\phi_i\rangle\langle\phi_i|\Big\|_1\left(\|\hat{\boldsymbol{\rho}}_i\|_1\Big\||\psi_i\rangle\langle\psi_i|\Big\|_1+\Big\||\phi_i\rangle\langle\phi_i|\Big\|_1\|\hat{\boldsymbol{\rho}}_i\|_1\right)\leq \tag{27}$$

$$2\Big\||\psi_i\rangle\langle\psi_i|-|\phi_i\rangle\langle\phi_i|\Big\|_1=2\sqrt{1-|\langle\psi_i|\phi_i\rangle|^2}= \tag{28}$$

$$2\sqrt{1-\left|\frac{1}{\alpha_i}\left(1-\sum_{k=1}^{i-1}|\langle\phi_k|\psi_i\rangle|^2\right)\right|^2}=2\sqrt{1-\left|\frac{\left(1-\sum_{k=1}^{i-1}|\langle\phi_k|\psi_i\rangle|^2\right)}{\sqrt{\left(1-\sum_{k=1}^{i-1}|\langle\phi_k|\psi_i\rangle|^2\right)}}\right|^2} \tag{29}$$

$$=2\sqrt{1-1+\sum_{k=1}^{i-1}|\langle\phi_k|\psi_i\rangle|^2}=2\sqrt{\sum_{k=1}^{i-1}|\langle\phi_k|\psi_i\rangle|^2}\leq 2\sum_{k=1}^{i-1}|\langle\phi_k|\psi_i\rangle| \tag{30}$$

where we have used the fact that $\sum_{k=1}^{i-1}|\langle\phi_k|\psi_i\rangle|^2\leq 1$ in the last line (*Bessel's* inequality). □

The term $\sum_{k=1}^{i-1}|\langle\phi_k|\psi_i\rangle|$ may be understood by analyzing it through the scope of its related *Gram Determinant*. We present this as a lemma.

Lemma 3.

$$\left|\phi_j\right\rangle = \frac{1}{\sqrt{D_{j-1}D_j}} \begin{vmatrix} \left\langle\psi_1\middle|\psi_1\right\rangle & \left\langle\psi_1\middle|\psi_2\right\rangle & \dots & \left\langle\psi_1\middle|\psi_j\right\rangle \\ \left\langle\psi_2\middle|\psi_1\right\rangle & \left\langle\psi_2\middle|\psi_2\right\rangle & \dots & \left\langle\psi_2\middle|\psi_j\right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle\psi_{j-1}\middle|\psi_1\right\rangle & \left\langle\psi_{j-1}\middle|\psi_2\right\rangle & \dots & \left\langle\psi_{j-1}\middle|\psi_j\right\rangle \\ \left|\psi_1\right\rangle & \left|\psi_2\right\rangle & \dots & \left|\psi_j\right\rangle \end{vmatrix} \tag{31}$$

where

$$D_j := \begin{vmatrix} \left\langle\psi_1\middle|\psi_1\right\rangle & \left\langle\psi_1\middle|\psi_2\right\rangle & \dots & \left\langle\psi_1\middle|\psi_j\right\rangle \\ \left\langle\psi_2\middle|\psi_1\right\rangle & \left\langle\psi_2\middle|\psi_2\right\rangle & \dots & \left\langle\psi_2\middle|\psi_j\right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle\psi_j\middle|\psi_1\right\rangle & \left\langle\psi_j\middle|\psi_2\right\rangle & \dots & \left\langle\psi_j\middle|\psi_j\right\rangle \end{vmatrix} \tag{32}$$

with the definitions $\left|\phi_1\right\rangle := \left|\psi_1\right\rangle$, $D_0 := 1$ and $D_1 = 1$; making consistent the case where $j = 1$ and $k = 0, 1$ for $\left|\phi_i\right\rangle$ and D_k respectively. The vertical lines to the left and to the rigth of the above arrays indicate that a determinant is being taken.

In determinant form, $\left\langle\psi_i\middle|\phi_k\right\rangle$ may now be written as follows.

$$\left\langle\psi_i\middle|\phi_k\right\rangle = \frac{1}{\sqrt{D_{k-1}D_k}} \begin{vmatrix} \left\langle\psi_1\middle|\psi_1\right\rangle & \left\langle\psi_1\middle|\psi_2\right\rangle & \dots & \left\langle\psi_1\middle|\psi_k\right\rangle \\ \left\langle\psi_2\middle|\psi_1\right\rangle & \left\langle\psi_2\middle|\psi_2\right\rangle & \dots & \left\langle\psi_2\middle|\psi_k\right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle\psi_{k-1}\middle|\psi_1\right\rangle & \left\langle\psi_{k-1}\middle|\psi_2\right\rangle & \dots & \left\langle\psi_{k-1}\middle|\psi_k\right\rangle \\ \left\langle\psi_i\middle|\psi_1\right\rangle & \left\langle\psi_i\middle|\psi_2\right\rangle & \dots & \left\langle\psi_i\middle|\psi_k\right\rangle \end{vmatrix} \tag{33}$$

The power of viewing the states $\left|\phi_i\right\rangle$ in their determinant form is that now we need only compute inner products between elements of the set $\left\{\left|\psi_i\right\rangle\right\}_{i=1}^{N}$ in order to estimate the effectiveness of the *PVM* (18) in approximating a solution to the SQSD problem with PVM, i.e. $\min_{PVM} \sum_{i=1}^{N} p_i \left\| \hat{\boldsymbol{\rho}}_i - \hat{\mathbf{P}}_i \hat{\boldsymbol{\rho}}_i \hat{\mathbf{P}}_i \right\|_1$. Recall that the states $\left\{\left|\psi_i\right\rangle\right\}_{i=1}^{N}$ are normalized and let us consider the case where $\left\langle\psi_i\middle|\psi_j\right\rangle = \varepsilon_{ij}$ for all $i \neq j \in \{1, ..., N\}$, where ε_{ij} are complex numbers satisfying $|\varepsilon_{ij}| \leq \delta$ for all $i \neq j \in \{1, .., N\}$, where δ is small. Since, under this assumption, all entries of the last column of the matrix (33) are small, this would also imply that $\left\| \hat{\boldsymbol{\rho}}_i - |\phi_i\rangle\langle\phi_i| \hat{\boldsymbol{\rho}}_i |\phi_i\rangle\langle\phi_i| \right\|_1$ is small for all i, thanks to Lemma 2.

The above estimates imply the following theorem.

Theorem 3. *Consider a mixed state of the form $\sum_{i=1}^{N} p_i \hat{\boldsymbol{\rho}}_i$, $\sum_{i=1}^{N} p_i = 1$, where $\hat{\boldsymbol{\rho}}_i := \left|\psi_i\right\rangle\left\langle\psi_i\right|$ are pure states acting on a* Hilbert *space of dimension greater than N. Furthermore, assume that the states $\{\left|\psi_i\right\rangle\}_i$ are linearly independent. Then*

$$\min_{POVM} \sum_{i=1}^{N} p_i \left\| \hat{\boldsymbol{\rho}}_i - \hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}}_i \hat{\mathbf{M}}_i^{\dagger} \right\|_1 \leq \sum_{i=2}^{N} p_i \sum_{k=1}^{i-1} \left| \frac{M_{k,i}}{D_{k-1} D_k} \right| \tag{34}$$

where

$$M_{k,i} := 2 \begin{vmatrix} \langle\psi_1|\psi_1\rangle & \langle\psi_1|\psi_2\rangle & \dots & \langle\psi_1|\psi_k\rangle \\ \langle\psi_2|\psi_1\rangle & \langle\psi_2|\psi_2\rangle & \dots & \langle\psi_2|\psi_k\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi_{k-1}|\psi_1\rangle & \langle\psi_{k-1}|\psi_2\rangle & \dots & \langle\psi_{k-1}|\psi_k\rangle \\ \langle\psi_i|\psi_1\rangle & \langle\psi_i|\psi_2\rangle & \dots & \langle\psi_i|\psi_k\rangle \end{vmatrix} \tag{35}$$

$$D_k := \begin{vmatrix} \langle\psi_1|\psi_1\rangle & \langle\psi_1|\psi_2\rangle & \dots & \langle\psi_1|\psi_k\rangle \\ \langle\psi_2|\psi_1\rangle & \langle\psi_2|\psi_2\rangle & \dots & \langle\psi_2|\psi_k\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi_k|\psi_1\rangle & \langle\psi_k|\psi_2\rangle & \dots & \langle\psi_k|\psi_k\rangle \end{vmatrix} \tag{36}$$

Proof. The proof follows directly from Lemma 3 and Lemma 2, and the fact that for $i = 1$ the corresponding projector is simply $|\psi_i\rangle\langle\psi_i|$ making the $i = 1$ term zero. □

It is with the bound provided by Theorem 3 that we may estimate the right-hand side of (13). Notice that the magnitudes of the elements of the determinants found in (34) and (36) are all bounded by the square root of the respective fidelities. i.e noting that [9]

$$\langle\psi_i|\psi_j\rangle \leq |\langle\psi_i|\psi_j\rangle| = \sqrt{F\Big(|\psi_i\rangle\langle\psi_i|, |\psi_j\rangle\langle\psi_j|\Big)}. \tag{37}$$

With the latter relationship, it is clear that the bound of Theorem 3 will consist purely of fidelities as was alluded to in the previous section.

3 Dynamical Monitoring for discrete variables

Although SBS are interesting objects of study without regard to anything else, more intriguing is studying the dynamical convergence of some time-dependent density operator $\hat{\boldsymbol{\rho}}_t$ to a SBS state. If we now apriori that a certain type of multifaceted quantum mechanical system should behave objectively per Definitions 1 and 2, then the states of said multifaceted systems should exhibit convergence to some SBS state in physically relevant time domains which often include the asymptotical case of $t \to \infty$, where t is the dynamical parameter in question. Time dependence may in general be generated by some arbitrary time-dependent quantum map. However, we will focus on the quantum maps generated from *quantum-measurement* limit type interaction Hamiltonians of the *von Neumann type* [18] which are central to the theory of *Quantum Decoherence* [30]. Before we get to the heart of the matter concerning these types of quantum maps, we shall define the concept of quantum-measurement limit and the von Neumann type Hamiltonains.

3.1 Quantum-Measurement Limit

The principal models studied in SBS literature [37][38][39] are of the *quantum-measurement* limit type, meaning SBS that arise from dynamics generated by Hamiltonians in which the interaction term between the system S and the environment E greatly dominates, i.e. $\hat{\mathbf{H}}_{tot} \approx \hat{\mathbf{H}}_{int}$ (tot means total and *int* indicates "interaction terms"). Such an approximation is valid when the system and the environments evolve with respect to a time scale that is much larger than that of the time scale corresponding to that of the interactive dynamics. In this work, we will furthermore narrow our focus to interaction Hamiltonains of the following form

$$\hat{\mathbf{H}}_{int} = \hat{\mathbf{X}} \otimes \sum_{k=1}^{N} g_k \hat{\mathbf{B}}_k \tag{38}$$

A Hamiltonian of the form (38) is said to be of the von Neumann type [18]. The corresponding time evolution operator is hence

$$\hat{\mathbf{U}}_t = e^{-it\hat{\mathbf{X}} \otimes \sum_{k=1}^{N} g_k \hat{\mathbf{B}}_k}. \tag{39}$$

The theory of SBS for discrete variables focuses on the case where the system S is described by a finite-dimensional Hilbert space [37][38][39]. As such the self-adjoint operator $\hat{\mathbf{X}}$ will have purely discrete spectrum, i.e. only eigenvalues. Let $\{|i\rangle\}_{i=1}^{d_S}$ be the set of eigenvectors of $\hat{\mathbf{X}}$ with corresponding eigenvalues x_i; d_S is the dimension of $\mathscr{H}_S$ the Hilbert space associated with the sytem S. $\hat{\mathbf{B}}_k$ will be assumed to be an arbitrary self-adjoint operator. We shall see in what is to come that the spectral properties of the operator $\hat{\mathbf{B}}$ will determine whether or not the multipartite states we shall be studying converge to an SBS state.

3.2 Partial Tracing

In what follows we will follow the approach taken in [39] and revert back to the notational conventions used in the first section. We consider a quantum system interacting with N macroscopic environments. We assume that the joint initial state has the product form:

$$\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\rho}}_{S_0} \otimes \bigotimes_{k=1}^{N} \hat{\boldsymbol{\rho}}^{E_0^k} \tag{40}$$

In the sate (40) we write the subscript 0 in E_0^k in order to emphasize that this is the initial state of the kth environment E^k, similarly, we use the subscript S_0 to highlight the initial state of the system. We evolve our total initial state using the evolution operator (39).

$$\hat{\boldsymbol{\rho}}_t = \left(e^{-it\hat{\mathbf{X}} \otimes \sum_{k=1}^{N} g_k \hat{\mathbf{B}}_k}\right) \hat{\boldsymbol{\rho}}_{S_0} \otimes \bigotimes_{k=1}^{N} \hat{\boldsymbol{\rho}}^{E_0^k} \left(e^{it\hat{\mathbf{X}} \otimes \sum_{k=1}^{N} g_k \hat{\mathbf{B}}_k}\right). \tag{41}$$

To study the state of the subsystem formed by the system S and the first N_E environments, we take the partial trace of the time-evolved density operator over the remaining $M_E := N - N_E$ environments. The result is,

$$\sum_{i,j=1}^{d_S} \sigma_{i,j} \Gamma(i,j,t) |i\rangle\langle j| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_{x_i,x_j}^{E_t^k} \tag{42}$$

where, again, $\{|i\rangle\}_{i=1}^{d_s}$ are the eigenvectors of $\hat{\mathbf{X}}$, with corresponding eigenvalues $\{x_i\}_{i=1}^{d_S}$ and we have the following definitions.

$$\hat{\boldsymbol{\rho}}_{x,y}^{E_t^k} := e^{-itxg_k\hat{\mathbf{B}}_k} \hat{\boldsymbol{\rho}}^{E_0^k} e^{itying_k\hat{\mathbf{B}}_k} \; (k = 1, 2, ..., N_E) \tag{43}$$

$$\hat{\boldsymbol{\rho}}_{x}^{E_t^k} := e^{-itxg_k\hat{\mathbf{B}}_k} \hat{\boldsymbol{\rho}}^{E_0^k} e^{itxg_k\hat{\mathbf{B}}_k} \; (k = 1, 2, ..., N_E). \tag{44}$$

$$\sigma_{i,j} := \langle i|\hat{\boldsymbol{\rho}}_{S_0}|j\rangle \tag{45}$$

$$\gamma^k_{i,j}(t) := Tr\{\hat{\boldsymbol{\rho}}^{E^k_t}_{x_i,x_j}\} \tag{46}$$

$$\Gamma(i,j,,t) := \prod_{n=N_E+1}^{N} \gamma^n_{i,j}(t) \tag{47}$$

Let us now define the following quantum map [9].

$$\Lambda\Big(\hat{\boldsymbol{\rho}}_{S_0} \otimes \bigotimes_{k=1}^{N} \hat{\boldsymbol{\rho}}^{E^k_0}\Big) := \sum_{i,j=1}^{d_S} \sigma_{i,j}\Gamma(i,j,t)|i\rangle\langle j| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}^{E^k_t}_{x_i,x_j} \tag{48}$$

The right-hand side of (48) may be compactly rewritten as

$$\mathscr{U}_t\Big(\mathscr{E}_t\{\hat{\boldsymbol{\rho}}_{S_0}\} \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}^{E^k_0}\Big) \tag{49}$$

where

$$\mathscr{U}_t(\hat{\mathbf{A}}) := e^{-it\hat{\mathbf{X}}\otimes\sum_{k=1}^{N_E} g_k\hat{\mathbf{B}}_k}(\hat{\mathbf{A}})e^{it\hat{\mathbf{X}}\otimes\sum_{k=1}^{N_E} g_k\hat{\mathbf{B}}_k} \tag{50}$$

and

$$\mathscr{E}_t(\hat{\mathbf{C}}) := \sum_{i,j=1}^{d_S} \langle i|\hat{\mathbf{C}}|j\rangle\Gamma(i,j,t)|i\rangle\langle j| \tag{51}$$

Deriving (48) from (49) is not very involved. One need only utilize the eigenvectors of $\hat{\mathbf{X}}$ in order to rewrite the density operator $\hat{\boldsymbol{\rho}}_{S_0}$ on the left-hand side of (48). The trace-preserving quantum map Λ is a composition of two trace-preserving quantum $\mathscr{U}_t$ and $\mathscr{E}_t$: a unitary map acting on S and the environmental degrees of freedom that were not traced out and a non-unitary map acting locally in S.

4 Monitoring the Process of System Information Broadcasting

In [39], a SBS state associated with (42) is defined for every value of $t > 0$; the goal therein was to show that (42) converges to an associated SBS state as t goes to ∞. The associated SBS state of (42) at time t is defined as follows. We first restrict the sum of (42) to the diagonal terms—the terms with $i = j$. We will label the resulting operator as follows.

$$\hat{\boldsymbol{\rho}}_{dg,t} = \sum_{i=1}^{d_S} \sigma_i|i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}^{E^k_t}_{x_i} \tag{52}$$

The next step is to choose for every t a PVM acting on the space $\mathcal{S}\big(\mathscr{H}_S \otimes \bigotimes_{k=1}^{N_E} \mathscr{H}_{E^k}\big)$ (Note that for the case considered in [39], $dim(\mathscr{H}_S) = d_S < \infty$ and $dim(\mathscr{H}_{E^k}) = d_{E^k} < \infty$ for all k). To define such a PVM, the authors use the eigenbasis of the operator $\hat{\mathbf{X}}$: the elements of the PVM are of the form $|i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}^{E^k_t}_j$ where the $\{|i\rangle\langle i|\}_{i=1}^{d_S}$ and $\{\hat{\mathbf{P}}^{E^k_t}_j\}_{j=1}^{d_S} \cup \{\mathbb{I} - \sum_{i=1}^{d_S} \hat{\mathbf{P}}^{E^k_t}_i\}$ resolve the identity operators in $\mathcal{B}(\mathscr{H}_S)$ and $\mathcal{B}(\mathscr{H}_{E_k})$ respectively, so that, in particular, $\{\hat{\mathbf{P}}^{E^k_t}_j\}_{j=1}^{d_S} \cup \{\mathbb{I} - \sum_{i=1}^{d_S} \hat{\mathbf{P}}^{E^k_t}_i\}$ is a PVM in the kth environment's Hilbert space. The latter PVMs are then used to approximate the operator (42) by an SBS state:

$$\hat{\boldsymbol{\rho}}_{SBS,t} := \frac{1}{\mathscr{N}} \sum_{j=1}^{d_S} \left(|j\rangle\langle j| \otimes \bigotimes_{k=1}^{N_E} \boldsymbol{P}_j^{E_t^k} \right) \hat{\boldsymbol{\rho}}_{diag,t} \left(|j\rangle\langle j| \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_j^{E_t^k} \right) = \tag{53}$$

$$\sum_{i=1}^{d_S} \tilde{\sigma}_i |i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E} \left(\hat{\mathbf{P}}_i^{E_t^k} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} \hat{\mathbf{P}}_i^{E_t^k} \right). \tag{54}$$

Here $\mathscr{N}$ is a normalizing constant and $\tilde{\sigma}_i := \frac{\sigma_i}{\mathscr{N}}$. With a bit of thought, one can verify that the operator (54) is indeed an SBS state as defined in Definition 2. If (42) converges to an object with the form (54) as $t \to \infty$, we say that (42) is asymptotically SBS. Convergence is meant here in the sense of trace distance. Namely, one would like to show that

$$\frac{1}{2} \min_{PVM} \left\| \hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{SBS,t} \right\|_1 \to 0 \text{ as } t \to \infty \tag{55}$$

where for each t the minimization is taken over all projective-valued-measures $\{\hat{\mathbf{P}}_i^{E_t^k}\}_{i=1}^{d_S} \cup \{\mathbb{I} - \sum_{i=1}^{d_S} \hat{\mathbf{P}}_i^{E_t^k}\}$. Utilizing the fact that $\frac{1}{2}\min_{POVM} \left\| \hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{SBS,t} \right\|_1 \leq \frac{1}{2}\min_{PVM} \left\| \hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{SBS,t} \right\|_1$ we may conclude that (55) implies that $\frac{1}{2}\min_{POVM} \left\| \hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{SBS,t} \right\|_1 \to 0$ as $t \to \infty$ as well. An attempt is made in [39] to prove (55) for he current setting but the argument provided there is incomplete. In what follows we discuss the bounds presented in [39], as well as propose and prove an alternative bound for the trace distance in (55).

In [39], a bound is conjectured for the trace distance in (55). In the case where M_E environmental degrees of freedom have been traced out and N_E remain, the bound looks as follows.

$$\frac{1}{2} \min_{PVM} \left\| \hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{SBS,t} \right\|_1 \leq \Gamma(t) + \sum_i \sum_{j;j\neq i} \sqrt{\sigma_i \sigma_j} \sum_{k=1}^{N_E} F\left(\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}, \hat{\boldsymbol{\rho}}_{x_j}^{E_t^k}\right) \tag{56}$$

where now, $\Gamma(t) := \sum_i \sum_{j;j\neq i} |\sigma_{i,j}| \prod_{k=N_E+1}^{N} |\gamma_{i,j}^k(t)|$, and again $\gamma_{i,j}^k(t) = Tr\left[\hat{\boldsymbol{\rho}}_{x_i,x_j}^{E_t^k}\right]$, $\sigma_{i,j} := \langle i|\hat{\boldsymbol{\rho}}_{S_0}|j\rangle$. If true, this result would allow us to estimate the minimum on the LHS, using the asymptotic properties of $\Gamma(t)$ and the fidelity terms in (56). This estimate would further give us a way to estimate $\frac{1}{2}\min_{POVM} \left\| \hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{SBS,t} \right\|_1$. As (56) is currently not known to be true, we will not be using it. Instead, we will be utilizing the bound proven in the previous section which constitutes Theorem 3. For the interested reader, we include a discussion pointing out the gap in the proof of the main theorem of [39] in appendix A.

4.1 A New Bound for the Trace Distance of a Multipartite State and an Approximating SBS state

In what follows we use an unnormalized version of (53): $\hat{\boldsymbol{\rho}}_{PSBS,t} := \mathscr{N}\hat{\boldsymbol{\rho}}_{SBS,t}$. This state is just (53) without the normalization factor $\frac{1}{\mathscr{N}}$. In practice it is easier to bound $\left\| \hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{PSBS,t} \right\|_1$ and then utilize Lemma 4, stated below, to bound $\left\| \hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{SBS,t} \right\|_1$.

Lemma 4. *$\left\| \hat{\boldsymbol{\rho}} - \eta\hat{\boldsymbol{\sigma}} \right\|_1 \leq L$ implies $\left\| \hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\sigma}} \right\| \leq 2L$ for constants $L \geq 0$ and $\eta \in [0,1]$*

Proof. Using reverse triangle inequality we see that

$$L \geq \|\hat{\boldsymbol{\rho}} - \eta\hat{\boldsymbol{\sigma}}\|_1 \geq \big|\|\hat{\boldsymbol{\rho}}\|_1 - \|\eta\hat{\boldsymbol{\sigma}}\|_1\big| = \|\hat{\boldsymbol{\rho}}\|_1 - \|\eta\hat{\boldsymbol{\sigma}}\|_1 = 1 - \eta \tag{57}$$

furthermore

$$\|\hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\sigma}}\|_1 = \|\hat{\boldsymbol{\rho}} - \eta\hat{\boldsymbol{\sigma}} + \eta\hat{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}}\|_1 \leq \|\hat{\boldsymbol{\rho}} - \eta\hat{\boldsymbol{\sigma}}\|_1 + \|\eta\hat{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}}\|_1 \leq \tag{58}$$

$$L + (1-\eta)\|\hat{\boldsymbol{\sigma}}\|_1 = L + (1-\eta) \leq L + L = 2L \tag{59}$$

□

We now prove a preliminary inequality.

$$\left\|\hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{PSBS,t}\right\|_1 = \tag{60}$$

$$\left\| \sum_{i,j=1}^{d_S} \sigma_{i,j}\Gamma(i,j,t)|i\rangle\langle j| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}^{E_t^k}_{x_i,x_j} - \sum_{i=1}^{d_S} \sigma_i |i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k} \hat{\boldsymbol{\rho}}^{E_t^k}_{x_i} \hat{\mathbf{P}}_i^{E_t^k} \right\|_1 \leq \tag{61}$$

$$\left\| \sum_{i=1}^{d_S} \sigma_i |i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}^{E_t^k}_{x_i} - \sum_{i=1}^{d_S} \sigma_i |i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k} \hat{\boldsymbol{\rho}}^{E_t^k}_{x_i} \hat{\mathbf{P}}_i^{E_t^k} \right\|_1 + \left\| \sum_i \sum_{j;j\neq i}^{d_S} \sigma_{i,j}\Gamma(i,j,t)|i\rangle\langle j| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}^{E_t^k}_{x_i,x_j} \right\|_k \leq \tag{62}$$

$$\sum_{i=1}^{d_S} \left\| \sigma_i |i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}^{E_t^k}_{x_i} - \sigma_i |i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k} \hat{\boldsymbol{\rho}}^{E_t^k}_{x_i} \hat{\mathbf{P}}_i^{E_t^k} \right\|_1 + \left\| \sum_i \sum_{j;j\neq i}^{d_S} \sigma_{i,j}\Gamma(i,j,t)|i\rangle\langle j| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}^{E_t^k}_{x_i,x_j} \right\|_1 \leq \tag{63}$$

$$\sum_{i=1}^{d_S} \sigma_i \left\| |i\rangle\langle i| \otimes \left(\bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}^{E_t^k}_{x_i} - \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k} \hat{\boldsymbol{\rho}}^{E_t^k}_{x_i} \hat{\mathbf{P}}_i^{E_t^k} \right) \right\|_1 + \sum_i \sum_{j;j\neq i}^{d_S} \left|\sigma_{i,j}\Gamma(i,j,t)\right| \left\| |i\rangle\langle j| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}^{E_t^k}_{x_i,x_j} \right\|_1 = \tag{64}$$

$$\sum_{i=1}^{d_S} \sigma_i \left\| \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}^{E_t^k}_{x_i} - \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k} \hat{\boldsymbol{\rho}}^{E_t^k}_{x_i} \hat{\mathbf{P}}_i^{E_t^k} \right\|_1 + \sum_i \sum_{j;j\neq i}^{d_S} \left|\sigma_{i,j}\Gamma(i,j,t)\right| \leq \tag{65}$$

$$\sum_{k=1}^{N_E} \sum_{i=1}^{d_S} \sigma_i \left\| \hat{\boldsymbol{\rho}}^{E_t^k}_{x_i} - \hat{\mathbf{P}}_i^{E_t^k} \hat{\boldsymbol{\rho}}^{E_t^k}_{x_i} \hat{\mathbf{P}}_i^{E_t^k} \right\|_1 + \sum_i \sum_{j;j\neq i}^{d_S} \left|\sigma_{i,j}\Gamma(i,j,t)\right| \tag{66}$$

Where in the last step we have used Lemma 1. Now, using Lemma 4 we conclude that

$$\frac{1}{2} \min_{PVM} \left\|\hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{SBS,t}\right\|_1 \leq \min_{PVM} \left(\sum_{k=1}^{N_E} \sum_{i=1}^{d_S} \sigma_i \left\| \hat{\boldsymbol{\rho}}^{E_t^k}_{x_i} - \hat{\mathbf{P}}_i^{E_t^k} \hat{\boldsymbol{\rho}}^{E_t^k}_{x_i} \hat{\mathbf{P}}_i^{E_t^1} \right\|_1 \right) + \Gamma(t) \tag{67}$$

Recalling that $\Gamma(t) := \sum_i \sum_{j;j\neq i} |\sigma_{i,j}| \prod_{k=N_E+1}^{N} |\gamma^k_{i,j}(t)|$, $\gamma^k_{i,j}(t) = Tr\big[\hat{\boldsymbol{\rho}}^{E_t^k}_{x_i,x_j}\big]$, and $\sigma_{i,j} := \langle i|\hat{\boldsymbol{\rho}}_{S_0}|j\rangle$. $\Gamma(t)$ (67) is the decoherence term which is independent of the choice of the PVM minimized over. The decoherence term is simple to study provided that we are able to compute the trace defining the terms $\gamma^k_{i,j}(t)$. The first term in (67) involves a minimization over all PVM for each value of t. Rather than attempting to solve the minimization problem exactly, we shall be employing Theorem 3 to bound (67).

In order to apply Theorem 3 to estimate the first term in (67) we must assume that the initial states $\hat{\boldsymbol{\rho}}^{E_0^k}$ are pure, we will consider the case where these are not pure in Section 5. The purity of $\hat{\boldsymbol{\rho}}^{E_0^k}$ furthermore implies that the operators $\hat{\boldsymbol{\rho}}^{E_t^k}_i$ are pure for all i since the evolution (44) is unitary.

We will henceforth write $\hat{\boldsymbol{\rho}}^{E_0^k}$ as a projector.

$$\left|\psi_{i,t}^k\right\rangle\left\langle\psi_{i,t}^k\right| = \hat{\boldsymbol{\rho}}_i^{E_t^k} \tag{68}$$

We now use Theorem 3 to estimate the first summand of (67), therefore leading the following theorem.

Theorem 4. *Using the definitions found in this section so far.*

$$\frac{1}{2}\min_{POVM}\left\|\hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{SBS,t}\right\|_1 \leq \frac{1}{2}\sum_{k=1}^{N_E}\sum_{i=2}^{d_S}\sigma_i\sum_{s=1}^{i-1}\left|\frac{M_{s,i}^k}{D_{s-1,t}^k D_{s,t}^k}\right| + \Gamma(t) \tag{69}$$

where

$$M_{s,i}^k := \begin{vmatrix} \langle\psi_{1,t}^k|\psi_{1,t}^k\rangle & \langle\psi_{1,t}^k|\psi_{2,t}^k\rangle & \dots & \langle\psi_{1,t}^k|\psi_{s,t}^k\rangle \\ \langle\psi_{2,t}^k|\psi_{1,t}^k\rangle & \langle\psi_{2,t}^k|\psi_{2,t}^k\rangle & \dots & \langle\psi_{2,t}^k|\psi_{s,t}^k\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi_{s-1,t}^k|\psi_{1,t}^k\rangle & \langle\psi_{s-1,t}^k|\psi_{2,t}^k\rangle & \dots & \langle\psi_{s-1,t}^k|\psi_{s,t}^k\rangle \\ \langle\psi_{i,t}^k|\psi_{1,t}^k\rangle & \langle\psi_{i,t}^k|\psi_{2,t}^k\rangle & \dots & \langle\psi_{i,t}^k|\psi_{s,t}^k\rangle \end{vmatrix} \tag{70}$$

$$D_{s,t}^k := \begin{vmatrix} \langle\psi_{1,t}^k|\psi_{1,t}^k\rangle & \langle\psi_{1,t}^k|\psi_{2,t}^k\rangle & \dots & \langle\psi_{1,t}^k|\psi_{s,t}^k\rangle \\ \langle\psi_{2,t}^k|\psi_{1,t}^k\rangle & \langle\psi_{2,t}^k|\psi_{2,t}^k\rangle & \dots & \langle\psi_{2,t}^k|\psi_{s,t}^k\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi_{j,t}^k|\psi_{1,t}^k\rangle & \langle\psi_{j,t}^k|\psi_{2,t}^k\rangle & \dots & \langle\psi_{s,t}^k|\psi_{s,t}^k\rangle \end{vmatrix} \tag{71}$$

It is with Theorem 4 that we hope to mitigate the gap in [39]. Although Corollary 1 of [39] is not substantiated by a correct proof at the moment, we present Theorem 4 as a viable alternative to corollary 1 of [39]. If fate should have it that corollary 1 is shown to be fundamentally untrue, then Theorem 4 would be the only tool for us to choose from (i.e. to the extent of the author's knowledge).

5 Mixed Environmental States

Recall that we named the sums over i in (67) the SQSD problem for the mixture $\sum_i \sigma_i \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}$. We remind the reader that we call it Super Quantum State Discrimination (SQSD) because (67) bounds the respective QSD error $p_E\{p_i,\ \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k},\ \hat{\mathbf{P}}_i^{E_t^k}\}$ as follows.

$$p_E\Big\{p_i,\ \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k},\ \hat{\mathbf{P}}_i^{E_t^k}\Big\} = \sum_{i=1}^{d_S}\sigma_i Tr\Big\{\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} - \hat{\mathbf{P}}_i^{E_t^k}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\hat{\mathbf{P}}_i^{E_t^k}\Big\} \leq \sum_{i=1}^{d_S}\sigma_i\left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} - \hat{\mathbf{P}}_i^{E_t^k}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\hat{\mathbf{P}}_i^{E_t^k}\right\|_1 \tag{72}$$

where we have used the fact that $\left|Tr\{\hat{\mathbf{A}}\}\right| \leq \|\hat{\mathbf{A}}\|_1$.

The theory we have developed so far considers only the case where $\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}$ are pure states for all i and k. In this section, we will further develop the previous section by providing the analog to Theorem 3 for the case where the environmental degrees of freedom are finite mixtures of pure states. Using a simpler indexing scheme, consider a mixed state of the form $\sum_{i=1}^N p_i\hat{\boldsymbol{\rho}}_i$, where $\sum_{i=1}^N p_i = 1$ and the $\hat{\boldsymbol{\rho}}_i$ are all countably-mixed states; i.e. $\hat{\boldsymbol{\rho}}_i = \sum_{k=1}^{M_i}\eta_{ik}\hat{\boldsymbol{\rho}}_{ik}$ where all of the $\hat{\boldsymbol{\rho}}_{ik}$ are pure states and

$\sum_{k=1}^{M_i}\eta_{ik}=1$. Let us now consider the QSD problem

$$\min_{POVM}\sum_{i=1}^{N}p_iTr\Big\{\hat{\boldsymbol{\rho}}_i-\hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{M}}_i^{\dagger}\Big\}. \tag{73}$$

The latter item is bounded above by the minimization problem that we have been concerned with in the previous section, i.e. minimizing over all PVM as opposed to minimizing over all POVM in (73). In turn, it is also bounded above by the super PVM quantum state discrimination error as seen in the following relationship.

$$\min_{POVM}\sum_{i=1}^{N}p_iTr\Big\{\hat{\boldsymbol{\rho}}_i-\hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{M}}_i^{\dagger}\Big\}\leq\min_{PVM}\sum_{i=1}^{N}p_iTr\Big\{\hat{\boldsymbol{\rho}}_i-\hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{M}}_i^{\dagger}\Big\}\leq \tag{74}$$

$$\min_{PVM}\sum_{i=1}^{N}p_i\Big\|\hat{\boldsymbol{\rho}}_i-\hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{M}}_i^{\dagger}\Big\|_1 \tag{75}$$

As mentioned before, the latter follows from the fact that all PVMs are POVMs, making the space over which the objective function is minimized smaller and therefore yielding a smaller minimum.

Using the bound of Theorem 1 we will bound (74) and (75) from below. Namely,

$$\frac{1}{2}\sum_{i=1}^{N}\sum_{j;j\neq i}^{N}p_ip_jF(\hat{\boldsymbol{\rho}}_i,\hat{\boldsymbol{\rho}}_j)\leq\min_{PVM}\sum_{i=1}^{N}p_i\Big\|\hat{\boldsymbol{\rho}}_i-\hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}_{i=1}\hat{\mathbf{P}}_i\Big\|_1 \tag{76}$$

Expanding the $\hat{\boldsymbol{\rho}}_i$ we see that

$$F(\hat{\boldsymbol{\rho}}_i,\hat{\boldsymbol{\rho}}_j)=F\Big(\sum_{k=1}^{M_i}\eta_{ik}\hat{\boldsymbol{\rho}}_{ik},\sum_{k=1}^{M_j}\eta_{jk}\hat{\boldsymbol{\rho}}_{jk}\Big)\geq\sum_{k=1}^{\min\{M_i,M_j\}}\sqrt{\eta_{ik}\eta_{jk}}F\Big(\hat{\boldsymbol{\rho}}_{ik},\hat{\boldsymbol{\rho}}_{jk}\Big) \tag{77}$$

where we have used the joint concavity of the fidelity [9] in the last line of (77). (76) now implies that

$$\frac{1}{2}\sum_{i=1}^{N}\sum_{j;j\neq i}^{N}\sum_{k=1}^{\min\{M_i,M_j\}}p_ip_j\sqrt{\eta_{ik}\eta_{jk}}F\Big(\hat{\boldsymbol{\rho}}_{ik},\hat{\boldsymbol{\rho}}_{jk}\Big)\leq\min_{PVM}\sum_{i=1}^{N}p_i\Big\|\hat{\boldsymbol{\rho}}_i-\hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{P}}_i\Big\|_1 \tag{78}$$

This inequality shows that a necessary and sufficient condition for fully solving the SQSD optimization problem, i.e. for obtaining $\min_{PVM}\sum_{i=1}^{N}p_i\Big\|\hat{\boldsymbol{\rho}}_i-\hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{P}}_i\Big\|_1=0$, is that $\hat{\boldsymbol{\rho}}_{ik}\perp\hat{\boldsymbol{\rho}}_{jk}$ for all i,j,k where $i\neq j$. Otherwise, we run into the possibility of the SQSD being bounded away from zero by a significant amount. For the case where the $\hat{\boldsymbol{\rho}}_i$ are not mixed states the respective relationship is $\hat{\boldsymbol{\rho}}_i\perp\hat{\boldsymbol{\rho}}_j$ for $i\neq j$, which is what we expect from our analysis in the previous section. For the case where the $\hat{\boldsymbol{\rho}}_i$ are finite mixtures of pure states it is perhaps not surprising that one will be required to analyze the fidelities between elements of any two different mixtures, say $\hat{\boldsymbol{\rho}}_i$ and $\hat{\boldsymbol{\rho}}_j$, in order to determine the discriminability of the mixture $\sum_{i=1}^{N}\hat{\boldsymbol{\rho}}_i$. As informative as (78) is, we have yet to learn anything about the necessary constraints for the fidelities involving multiple elements of the same mixture $\hat{\boldsymbol{\rho}}_i$, take $\hat{\boldsymbol{\rho}}_{ik}$ and $\hat{\boldsymbol{\rho}}_{il}$ for example. It could be the case that, in principle, there need not be any restrictions on said fidelities in order to successfully achieve $\min_{PVM}\sum_{i=1}^{N}p_i\Big\|\hat{\boldsymbol{\rho}}_i-\hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{P}}_i\Big\|_1=0$; at the moment, however, the latter is unknown to the author.

We will now be bounding (78) from above. To do this, we will once again take a constructive approach. Our approach shall be an adaptation of the methods employed in the proof of Theorem 3 and Lemma 2. Constraining ourselves to the case where $\hat{\mathbf{P}}_i$ are projectors will yield a bound that will be useful for the cases where $\hat{\boldsymbol{\rho}}_{ik} \perp \hat{\boldsymbol{\rho}}_{jk}$ for all k when $i \neq j$ and $\hat{\boldsymbol{\rho}}_{ik} \perp \hat{\boldsymbol{\rho}}_{il}$ for all i when $l \neq k$ hold exactly and/or approximately.

Let us now construct a PVM that attempts to solve the SQSD problem on the right-hand side of inequality (76). We begin by noting that

$$\min_{PVM} \sum_{i=1}^{N} p_i \left\| \hat{\boldsymbol{\rho}}_i - \hat{\mathbf{P}}_i \hat{\boldsymbol{\rho}}_i \hat{\mathbf{P}}_i \right\|_1 \leq \min_{PVM} \sum_{i=1}^{N} \sum_{k=1}^{M_i} p_i \eta_{ik} \left\| \hat{\boldsymbol{\rho}}_{ik} - \hat{\mathbf{P}}_i \hat{\boldsymbol{\rho}}_{ik} \hat{\mathbf{P}}_i \right\|_1 \tag{79}$$

This looks very similar to the PVM QSD problem for pure states we tackled in the previous two sections (note that $p_i \eta_{ik}$ is a probability distribution) with the exception that now each element of the PVM $\{\hat{\mathbf{P}}_i\}_i$corresponds to all elements $\hat{\boldsymbol{\rho}}_{ik}$ of the ith element of the kth mixture. Following the methods from the previous section, one might suggest implementing the gram-schmidt procedure once more in order to obtain an orthonormal set of vectors $|\phi_i\rangle$, one for each i. However, in this case, the operators $\hat{\boldsymbol{\rho}}_i$ are mixed and therefore do not have a representation as a vector in the corresponding Hilbert space; being able to view the mixture $\sum_i p_i \hat{\boldsymbol{\rho}}_i$ as a single index ensemble of pure states was one of the key assumptions that lead to Theorem 3. Perhaps there is a way to implement the Gram-Schmidt process with the end of producing an analog to Theorem 3 in greater generality by using the Hilbert-Schmidt inner product; in particular for the case where all of the $\hat{\boldsymbol{\rho}}_i$ are mixtures. Unfortunately, the authors are unaware of any such approaches that have been met with success as of yet.

We now impose the following structural assumption on the $\hat{\mathbf{P}}_i$.

$$\hat{\mathbf{P}}_i = \sum_{k=1}^{M_i} \hat{\mathbf{P}}_{ik} \tag{80}$$

Now, in order to guarantee that a sum such as $\sum_{k=1}^{M} \hat{\mathbf{P}}_{ik}$ is a projector, we will need to assume that the $\hat{\mathbf{P}}_{ik}$ are all projectors with non-overlapping support.

Proof.

$$\hat{\mathbf{P}}_i^2 = \Big(\sum_{k=1}^{M_i} \hat{\mathbf{P}}_{ik}\Big)^2 = \sum_{k=1}^{M_i} \sum_{p=1}^{M_i} \hat{\mathbf{P}}_{ik} \hat{\mathbf{P}}_{ip} = \sum_{k=1}^{M_i} \sum_{p=1}^{M_i} \hat{\mathbf{P}}_{ik} \hat{\mathbf{P}}_{ip} \delta_{kp} = \sum_{k=1}^{M_i} \hat{\mathbf{P}}_{ik} = \hat{\mathbf{P}}_i \tag{81}$$

□

Since all of the $\hat{\boldsymbol{\rho}}_{ik}$ are pure states, we may apply the Gram-schmidt process in order to construct a PVM $\{\hat{\mathbf{P}}_{ik}\}_{ik}$. The resulting PVM elements $\hat{\mathbf{P}}_{ik}$, with the inclusion of the completion element $\mathbb{I} - \sum_i \sum_k \hat{\mathbf{P}}_{ik}$, form a PVM that resolves the identity. There are $N \times M$ states $\hat{\boldsymbol{\rho}}_{ik}$ since the index i ranges from 1 to N and the index k from 1 to M. Let us now visualize the set of these operators $\hat{\boldsymbol{\rho}}_{ik}$ as a vector as follows.

$$\vec{\mathscr{V}} := \begin{pmatrix} \hat{\boldsymbol{\rho}}_{11} & \dots & \hat{\boldsymbol{\rho}}_{1M_1} & \hat{\boldsymbol{\rho}}_{21} & \dots & \hat{\boldsymbol{\rho}}_{2M_2} & \dots & \hat{\boldsymbol{\rho}}_{N1} & \dots\dots & \hat{\boldsymbol{\rho}}_{NM_N} \end{pmatrix}. \tag{82}$$

Let us now do a relabeling and name the sth component of this vector $\mathscr{V}_s := \big|\xi_s\big\rangle\big\langle\xi_s\big|$. Given a specific value $s \in \{1, 2, ..., \sum_{i=1}^{N} M_i\}$ we can use the following formula to obtain the corresponding $\hat{\boldsymbol{\rho}}_{ik}$.

$$\big|\xi_s\big\rangle\big\langle\xi_s\big| := \hat{\boldsymbol{\rho}}_{f(s),g(s)} \tag{83}$$

where $f(x) := \lceil \frac{x}{M} \rceil$ and $g(x) := x \; modM$. Assuming that the $\big|\xi_s\big\rangle\big\langle\xi_s\big|$ form a linearly independent set we now apply the Gram-Schmidt process to obtain the family of orthonormal states

$$\big|\phi_1\big\rangle := \big|\xi_1\big\rangle \tag{84}$$

$$\big|\phi_s\big\rangle := \frac{1}{\alpha_s}\Big\{\big|\xi_s\big\rangle - \sum_{k=1}^{s-1}\big\langle\phi_k\big|\xi_s\big\rangle\big|\phi_k\big\rangle\Big\}, \;\; s \in \{1, 2, ..., \sum_{i=1}^{N} M_i\} \tag{85}$$

and as before $\alpha_i := \big\|\big|\xi_i\big\rangle - \sum_{k=1}^{i-1}\big\langle\phi_k\big|\xi_i\big\rangle\big|\phi_k\big\rangle\big\| = \sqrt{1 - \sum_{k=1}^{i-1}|\big\langle\phi_k\big|\xi_i\big\rangle|^2}$ for $i > 1$ and $\alpha_1 = 1$ are the respective normalization constants. An identity resolving PVM $\Big\{\big|\xi_s\big\rangle\big\langle\xi_s\big|\Big\}_s \bigcup \Big\{\mathbb{I} - \sum_s \big|\xi_s\big\rangle\big\langle\xi_s\big|\Big\}$ has thus been constructed, defining $\omega_s := p_{f(s)}\eta_{f(s),g(s)}$ we may now rewrite and bound

$$\sum_{i=1}^{N}\sum_{k=1}^{M_i} p_i\eta_{ik}\Big\|\hat{\boldsymbol{\rho}}_{ik} - \mathbf{P}_i\hat{\boldsymbol{\rho}}_{ik}\mathbf{P}_i\Big\|_1 \tag{86}$$

as follows.

$$\sum_s \omega_s \Big\|\big|\xi_s\big\rangle\big\langle\xi_s\big| - \Big(\sum_{l=f(s)}^{f(s)+M_{f(s)}}\big|\phi_l\big\rangle\big\langle\phi_l\big|\Big)\big|\xi_s\big\rangle\big\langle\xi_s\big|\Big(\sum_{l=f(s)}^{f(s)+M_{f(s)}}\big|\phi_l\big\rangle\big\langle\phi_l\big|\Big)\Big\|_1 = \tag{87}$$

$$\sum_s \omega_s \Big\|\big|\xi_s\big\rangle\big\langle\xi_s\big| - \big|\phi_s\big\rangle\big\langle\phi_s\big|\xi_s\big\rangle\big\langle\xi_s\big|\phi_s\big\rangle\big\langle\phi_s\big| - \Big(\sum_{l=f(s);l\neq s}^{f(s)+M_{f(s)}}\big|\phi_l\big\rangle\big\langle\phi_l\big|\Big)\big|\xi_s\big\rangle\big\langle\xi_s\big|\Big(\sum_{l=f(s);l\neq s}^{f(s)+M_{f(s)}}\big|\phi_l\big\rangle\big\langle\phi_l\big|\Big)\Big\|_1 \leq \tag{88}$$

$$\sum_s \omega_s \Big\|\big|\xi_s\big\rangle\big\langle\xi_s\big| - \big|\phi_s\big\rangle\big\langle\phi_s\big|\xi_s\big\rangle\big\langle\xi_s\big|\phi_s\big\rangle\big\langle\phi_s\big|\Big\|_1 + \sum_s \omega_s \Big\|\Big(\sum_{l=f(s);l\neq s}^{f(s)+M_{f(s)}}\big|\phi_l\big\rangle\big\langle\phi_l\big|\Big)\big|\xi_s\big\rangle\big\langle\xi_s\big|\Big(\sum_{l=f(s);l\neq s}^{f(s)+M_{f(s)}}\big|\phi_l\big\rangle\big\langle\phi_l\big|\Big)\Big\|_1 \leq \tag{89}$$

$$\sum_s \omega_s 2\sum_{k=1}^{s-1}|\big\langle\phi_k\big|\xi_s\big\rangle| + \sum_s \omega_s \sum_{l=f(s)\rceil;l\neq s}^{f(s)+M_{f(s)}}\sum_{k=f(s);k\neq s}^{f(s)+M_{f(s)}}\Big\|\big|\phi_l\big\rangle\big\langle\phi_l\big|\xi_s\big\rangle\big\langle\xi_s\big|\phi_k\big\rangle\big\langle\phi_k\big|\Big\|_1 = \tag{90}$$

$$\sum_s \omega_s 2\sum_{k=1}^{s-1}|\big\langle\phi_k\big|\xi_s\big\rangle| + \sum_s \omega_s \sum_{l=f(s);l\neq s}^{f(s)+M_{f(s)}}\sum_{k=f(s);k\neq s}^{f(s)+M_{f(s)}}|\big\langle\phi_l\big|\xi_s\big\rangle\big\langle\xi_s\big|\phi_k\big\rangle| \tag{91}$$

where we have used Lemma 2 in going from (89) to (90). Using Lemma 3 we may explicitly write the terms $|\big\langle\phi_l\big|\xi_s\big\rangle|$ as Gram-Schmidt determinants and use these to estimate the efficacy of the PVM built from (98).

In this work, mixed environmental states as the states of the environmental degrees of freedom are not the central focus. We shall therefore forego further analysis of the bound (91) at the moment and leave this for future work. However, we will point out that (91) may be further bounded by the following term.

$$(91) \leq 3\sum_s \omega_s \sum_{l;l\neq s}|\big\langle\phi_l\big|\xi_s\big\rangle| \tag{92}$$

where the only restriction on the sums is that $l \neq s$. As already mentioned, this may be better estimated using Lemma 3. We state the result (91) as a theorem.

Theorem 5. *Consider a mixed state of the form* $\sum_{i=1}^{N} p_i \hat{\boldsymbol{\rho}}_i$, $\sum_{i=1}^{N} p_i = 1$, *where* $\hat{\boldsymbol{\rho}}_i$ *are all countably-mixed states,i.e.* $\hat{\boldsymbol{\rho}}_i = \sum_{k=1}^{M_i} \eta_{ik} |\psi_{ik}\rangle\langle\psi_{ik}|$ *where all of the* $|\psi_{ik}\rangle\langle\psi_{ik}|$ *are pure states acting in a* Hilbert *space of dimension greater than* N *and* $\sum_{k=1}^{M_i} \eta_{ik} = 1$. *Furthermore, assume that the set* $\{|\psi_{ik}\rangle\}_{ik}$ *is linearly independent. Then*

$$\min_{POVM} \sum_{i=1}^{N} p_i \left\| \hat{\boldsymbol{\rho}}_i - \hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}}_i \hat{\mathbf{M}}_i^\dagger \right\|_1 \leq \tag{93}$$

$$2\sum_{s=1}^{S} \omega_s \sum_{k=1}^{s-1} |\langle\phi_k|\xi_s\rangle| + \sum_{s=1}^{S} \omega_s \sum_{l=f(s);l\neq s}^{f(s)+M_{f(s)}} \sum_{k=f(s);k\neq s}^{f(s)+M_{f(s)}} |\langle\phi_l|\xi_s\rangle\langle\xi_s|\phi_k\rangle| \tag{94}$$

where

$$|\xi_s\rangle := |\psi_{f(s),g(s)}\rangle\langle\psi_{f(s),g(s)}| \tag{95}$$

$$f(x) := \left\lceil \frac{x}{M} \right\rceil \tag{96}$$

$$g(x) := x \; modM \tag{97}$$

$$|\phi_1\rangle := |\xi_1\rangle \tag{98}$$

$$|\phi_s\rangle := \frac{1}{\alpha_s}\left\{ |\xi_s\rangle - \sum_{k=1}^{s-1} \langle\phi_k|\xi_s\rangle|\phi_k\rangle \right\}, \;\; s \in \{1, 2, ..., S := \sum_{i=1}^{N} M_i\} \tag{99}$$

and

$$\alpha_i := \left\| |\xi_i\rangle - \sum_{k=1}^{i-1} \langle\phi_k|\xi_i\rangle|\phi_k\rangle \right\| = \sqrt{1 - \sum_{k=1}^{i-1} |\langle\phi_k|\xi_i\rangle|^2} \tag{100}$$

for $i > 1$ *and* $\alpha_1 = 1$ *are the respective normalization constants.*

Proof. The proof may be found in the preceding discussion. □

As a final remark, we point out that a more general mixture of non-pure states $\sum_i p_i \hat{\boldsymbol{\rho}}_i$ may be obtained by considering the case where $\boldsymbol{\rho}_i := \mathscr{E}_i(\hat{\boldsymbol{\rho}}_0)$ ($\hat{\boldsymbol{\rho}}_0$ is a pure state); $\mathscr{E}_i$ being arbitrary quantum maps for all i. In general the $\boldsymbol{\rho}_i := \mathscr{E}_i(\hat{\boldsymbol{\rho}}_\mathbf{0})$ will not be expressable as finite mixtures. If such a case is encountered we may use (91) only if the $\boldsymbol{\rho}_i := \mathscr{E}_i(\hat{\boldsymbol{\rho}}_\mathbf{0})$ may be approximated by countable mixtures. In Chapter 5 we will study a case where a mixture of the type $\sum_i p_i \mathscr{E}_i(\hat{\boldsymbol{\rho}}_\mathbf{0})$ is encountered. However, for some of the cases to be studied in Chapter 5, the $\mathscr{E}_i$ will be approximately unitary maps and so we use this to approximate $\sum_i p_i \mathscr{E}_i(\hat{\boldsymbol{\rho}}_0)$ with a countable mixture of pure states. More general cases are still open to further investigation.

6 How general may the $\hat{\mathbf{B}}_k$ be?

We conclude this subsection with the following corollary. The question it sheds light on is the following: "How general may $\hat{\mathbf{B}}$ whilst still inducing dynamics (48) which are convergent to and SBS state?"

Theorem 6 (Necessary conditions for the convergence to SBS for a broad family of multipartite states)**.** *Consider the setup spanning equations (40) through (48). If for all k, $\hat{\mathbf{B}}_k$ has a non-empty Rajchman subspace $\mathscr{H}_{E^k,rc}$ ([58]), and $\hat{\boldsymbol{\rho}}_0^{E^k}$ is a finite mixture of pure states in $\mathcal{S}(\mathscr{H}_{E^k,rc})$, then $\hat{\boldsymbol{\rho}}$ converges asymptotically in $t > 0$ to an SBS state with respect to the trace norm topology.*

Proof. From the discussion in the previous section, which uses the techniques of Theorems 3 and 4, we see that the decay of the term $\sum_{k=1}^{N_E}\sum_{i=1}^{d_S}\sigma_i\left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} - \hat{\mathbf{P}}_i^{E_t^k}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\hat{\mathbf{P}}_i^{E_t^k}\right\|_1$ may be estimated via inner products of non-equal pure states, i.e. $\left|\langle\phi_l|\xi_s\rangle\right|$ (92) which upon inspecting its associated Gram-Determinants (see (36)) one should notice that the rightmost column has solely inner products of unequal pure states. Furthermore, the term $\Gamma(i,j,t)$ (47) is a product of inner products of non-equal pure states. The inner products emanating from both the diagonal terms $\sum_{k=1}^{N_E}\sum_{i=1}^{d_S}\sigma_i\left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} - \hat{\mathbf{P}}_i^{E_t^k}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\hat{\mathbf{P}}_i^{E_t^k}\right\|_1$ and the off-diagonal terms $\sum_i\sum_{j;j\neq i}^{d_S}\left|\sigma_{i,j}\Gamma(i,j,t)\right|$ and the diagonal terms are structural of the form $\langle\psi|e^{-it\hat{\mathbf{B}}_k}|\phi\rangle$ with $|\psi\rangle \in \mathscr{H}_{E^k,rc}$ and thus we have our result (using the fact that the Rajchman subspace is a reducing subsapce). ☐

A Quantum State Discrimination

Let $\{\hat{\mathbf{E}}_l\}_l$ be a POVM. The operators $\hat{\mathbf{E}}_l$ act over some unspecified Hilbert space. Indeed, $\sum_l \hat{\mathbf{E}}_l = \mathbb{I}$, $\|\hat{\mathbf{E}}_l\| \le 1$ and $\hat{\mathbf{E}}_l$ are positive operators that may be written as $\hat{\mathbf{E}}_l = \hat{\mathbf{M}}_l^\dagger\hat{\mathbf{M}}_l$ where $\hat{\mathbf{M}}_l$ are bounded operators. Now, let $\hat{\boldsymbol{\rho}}$ be a density operator acting over the same Hilbert space as the POVM $\{\hat{\mathbf{E}}_l\}_l$. A question arises regarding the positive semidefiniteness of the operator $\hat{\boldsymbol{\rho}} - \hat{\mathbf{M}}_l\hat{\boldsymbol{\rho}}\hat{\mathbf{M}}_l^\dagger$.

Claim 1 (Non positivity of a particular operator)**.** $\hat{\boldsymbol{\rho}} - \hat{\mathbf{M}}_l\hat{\boldsymbol{\rho}}\hat{\mathbf{M}}_l^\dagger$ *is not positive semidefinite in general*

Proof. Counter example.

Consider the 2 dimensional case where $\hat{\boldsymbol{\rho}} = \begin{pmatrix} 1-\delta & 0 \\ 0 & \delta \end{pmatrix}$ $(0 \le \delta \le 1)$ and we have a POVM characterized by the operator $\hat{\mathbf{M}}_0 = a\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$ $(a < 1)$ which is a scaled projector. The PVOM in question is

$$\left\{\hat{\mathbf{M}}_0^\dagger\hat{\mathbf{M}}_0,\ \mathbb{I} - \hat{\mathbf{M}}_0^\dagger\hat{\mathbf{M}}_0\right\}. \tag{101}$$

Let us take a look at the operator

$$\hat{\boldsymbol{\rho}} - \hat{\mathbf{M}}_0\hat{\boldsymbol{\rho}}\hat{\mathbf{M}}_0 \tag{102}$$

Expanding things out this looks as follows; in matrix notation.

$$\hat{\boldsymbol{\rho}} - \hat{\mathbf{M}}_0\hat{\boldsymbol{\rho}}\hat{\mathbf{M}}_0 = \begin{pmatrix} 1-\delta & 0 \\ 0 & \delta \end{pmatrix} - \frac{a^2}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1-\delta-\frac{a^2}{4} & -\frac{a^2}{4} \\ -\frac{a^2}{4} & \delta-\frac{a^2}{4} \end{pmatrix}. \tag{103}$$

For this operator to be positive semidefinite we require that $\langle\phi|\left\{\hat{\boldsymbol{\rho}} - \hat{\mathbf{M}}_0\hat{\boldsymbol{\rho}}\hat{\mathbf{M}}_0\right\}|\phi\rangle \ge 0$ hold for all $|\phi\rangle$ in the Hilbert space in question. Let us use the unit vector $\tilde{\mathbf{e}}_2 = (0,1)^t$. In this case

$$\langle\tilde{\mathbf{e}}|\left\{\hat{\boldsymbol{\rho}} - \hat{\mathbf{M}}_0\hat{\boldsymbol{\rho}}\hat{\mathbf{M}}_0\right\}|\tilde{\mathbf{e}}\rangle = \delta - \frac{a^2}{4} \tag{104}$$

But notice that if $\delta < \frac{a^2}{4}$, which is a viable possibility, then we do not have positive definiteness for $\hat{\boldsymbol{\rho}} - \hat{\mathbf{M}}_0\hat{\boldsymbol{\rho}}\hat{\mathbf{M}}_0$. □

That is it for the counter-example. Notice that in the case where $a = 1$, $\hat{\mathbf{M}}_0$ is a projector, and even then we do not have positive definiteness for $\hat{\boldsymbol{\rho}} - \hat{\mathbf{M}}_0\hat{\boldsymbol{\rho}}\hat{\mathbf{M}}_0$ in general since this breaks down for $\delta < \frac{1}{4}$.

Now, in the paper [39] the authors provide a proof for equation (5) on page 2 of said paper. This proof involves the computation of a trace distance of the form $\|\hat{\boldsymbol{\rho}} - \hat{\mathbf{P}}\hat{\boldsymbol{\rho}}\hat{\mathbf{P}}\|_1$ (where $\hat{\mathbf{P}}$ is a projector) ; see page (1) of the appendix of the same paper and look at the sentence preceding equation (4) of page one of this appendix. There the authors implicitly argue that $\|\hat{\boldsymbol{\rho}} - \hat{\mathbf{P}}\hat{\boldsymbol{\rho}}\hat{\mathbf{P}}\|_1 = Tr\Big\{\hat{\boldsymbol{\rho}}(\mathbb{I} - \hat{\mathbf{P}})\Big\}$ in general. This, however, is only true if $\hat{\boldsymbol{\rho}} - \hat{\mathbf{P}}\hat{\boldsymbol{\rho}}\hat{\mathbf{P}} \geq 0$, and this in turn is true only when $\hat{\mathbf{P}}$ commutes with $\hat{\boldsymbol{\rho}}$. It looks like, tacitly, they are assuming that the PVMS, amongst other assumptions, have the special property that (now I use their notation) the $\hat{\mathbf{P}}_i$ projector commute with the $\hat{\boldsymbol{\rho}}_i$ terms of the mixture $\sum_i p_i\hat{\boldsymbol{\rho}}_i$ where $\hat{\mathbf{P}}_i$ is an element of a POVM used to discriminate the mixture $\sum_i p_i\hat{\boldsymbol{\rho}}$. This assumption however need not in general be true and the bound by Knill and Barnum [26] does not assume commutativity for their result that bounds the trace

$$Tr\Big\{\sum_i p_i\hat{\boldsymbol{\rho}} - \sum_i \hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}\hat{\mathbf{M}}_i^\dagger\Big\} \tag{105}$$

to hold when minimizing over appropriate POVM, $\{\hat{\mathbf{M}}_i\}_i$, schemes and neither do they assume that we discriminate with projectors, their result uses the objective function which minimizes over all POVM. This means that the assumption that $\hat{\mathbf{P}}_i$ commutes with $\hat{\boldsymbol{\rho}}_i$ makes the minimization calculated in [39] an upper bound to the one proven by Knill and Barnum [26]. Unfortunately starting from $\|\hat{\boldsymbol{\rho}}_i - \hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{P}}_i\|_1$ and bounding such an object by fidelities is significantly harder

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