Linearly Dependent Sets of Algebraic Curvature Tensors

A CSUSB REU Report by Alberto Acevedo

Tuesday 16^{th} September, 2014 17:19

1 Abstract

We study the linear dependence properties of a set of three *canonical curvature tensor* $\{R_\varphi, R_\psi, R_\tau\}$ under different hypothesis which vary in the dimension of the vector space we are working with, the *ranks* of the *bilinear forms* τ and ψ, the *degeneracy* of φ, and a few other things such as diagonalization properties of the *bilinear forms* we are working with.

2 Introduction

Let V be a real vector space of finite dimension n and let $R \epsilon \otimes^4 V^*$. The *multilinear* function R is known as an *algebraic curvature tensor* if satisfies the following properties for all $w, x, y, z \epsilon V$:

$$R(x,y,z,w) = -R(y,x,z,w) \tag{1}$$

$$R(x,y,z,w) = R(z,w,x,y) \tag{2}$$

$$0 = R(x,y,z,w) + R(x,z,w,y) + R(x,w,y,z) \tag{3}$$

Definition 1:

Let $\mathcal{A}(V)$ be the vector space of all algebraic curvature tensors and let φ be a bilinear form on V. φ is *symmetric* if $\varphi(v,w) = \varphi(w,v)\ \forall\ v, w\epsilon V$ and positive definite if $\forall\ v\epsilon V\ \varphi(v,v) \geq 0$ were $\varphi(v,v) = 0$ only when $v = 0$. Let φ be a symmetric bilinear form on V. Define R_φ as follows.

$$R_\varphi(x,y,z,w) = \varphi(x,w)\varphi(y,z) - \varphi(x,z)\varphi(y,w) \tag{4}$$

It is simple to verify that $R_\varphi \in \mathcal{A}(V)$. It is known that $\mathcal{A}(V) = Span\{R_{\varphi_i} | \varphi_i is symmetric\}$ [1] . A natural question arises here, given $\varphi_1, \varphi_2, ..., \varphi_k, \in S^2(V)$ when is the set $R_{\varphi_1}, R_{\varphi_2}, ..., R_{\varphi_k}$ linearly independent?

Definition 2: Define $Spec(\psi)$ as the set of eigenvalues of ψ and $|Spec(\psi)|$ as the number of distinct elements of $Spec(\psi)$.

Definition 3: Given symmetric bilinear form Q, some basis vector e_i is said to be *spacelike* if $Q(e_i, e_i) > 0$ and *timelike* if $Q(e_i, e_i) < 0$.

Definition 4: If $\{e_1^-, ..., e_p^-, e_1^+, ...e_q^+\}$,were e_i^- are the timelike vectors and e_i^+ are the space like vectors,is a basis, then the basis is *orthonormal* if $Q(e_i^\pm, e_j^\pm) = \pm\delta_{ij}$ and $Q(e_i^-, e_j^+) = Q(e_i^+, e_j^-) = 0$.

Definition 5: A *symmetric bilinear* form Q with p timelike basis vectors and q spacelike basis vectors is said to have *signature* $SigQ = (p, q)$ which is independent of the *orthonormal* basis chosen.

Definition 6:The *Kernel* of a symmetric bilinear form Q is the set $kerQ = \{x \in V | Q(x, y) = 0\}$.

Theorem 1.1:[1] Suppose Rank $Rank\varphi \geq 3$. The Set $\{R_\varphi, R_\psi\}$ is linearly dependent iff $R_\psi \neq 0$, and $\varphi = \lambda\psi$ for some real number λ .

Theorem 1.2:[1] Suppose φ is positive definite, $Rank\tau = n$, and $Rank\psi \geq 3$. If $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent, then ψ and τ are simultaneously orthogonally diagonalizable with respect to φ

Theorem 1.3:[1] Suppose $dim(V) \geq 4$, φ is positive definite, $Rank\tau = n$, and $Rank\psi \geq 3$. The set $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent iff one of the following holds:

(1) $\mid Spec(\psi) \mid = \mid Spec(\tau) \mid = 1, or$

2)$Spec(\tau) = \{\eta_1, \eta_2, \eta_2, ...\}, and\ Spec(\psi) = \{\lambda_1, \lambda_2, \lambda_2, ...\}, with\ \eta_1 \neq \eta_2, \lambda_2^2 = \epsilon(\delta\eta_2^2 - 1)$,and $\lambda_1 = \frac{\epsilon}{\lambda_2}(\delta\eta_1\eta_2 - 1)$ for $\epsilon, \delta = \pm 1$.

Theorem 1.4: [6]A family of diagonalizable matrices is a commuting family iff it is a simultaneously diagonalizable family.
In this paper we shall focus on the linear dependence of three algebraic curva-

ture tensors. Throughout the paper, except for the final chapter, we assume that ψ and τ are symmetric bilinear forms in order that R_ψ and R_τ are both elements of $\mathcal{A}(V)$. Also, there will be times where we represent a bilinear form as a matrix and the following will illuminate the process.

Remark: A bilinear form Q with a basis $\{e_1, ..., e_n\}$ has a matrix representation were $Q(e_i, e_j) = a_{ij}$ is the component on the ith column and jth row.

Ex 1.6: Let Q be a bilinear form with a basis $\{e_1, e_2, e_3, e_4\}$ and $Q(e_i, e_j) = \delta_{ij}$. Then the following is how we represent Q as a matrix:

$$[Q] \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Just the identity 4×4 matrix.

3 Linear dependence of the set $\{R_\varphi, R_\tau, R_\psi\}$ were φ is ***nondegenerate***, $Rank\psi \geq 3$, $Rank\tau = dimV$, and $dimV \geq 4$.

We begin this section by stating the main theorem tobe proven but prove it for a specific case first ($k = 4$).

Theorem 2.1:Assume equation $R_\varphi = \varepsilon R_\psi + \delta R_\tau$ has a solution and let $\{e_1, ..., e_k, f_1, ..., f_k\}$ be an orthonormal basis (with respect to φ) of a vector space $V = V^+ \bigoplus V^-$ were $V^+ = span\{f_1, ..., f_k\}$ and $V^- = span\{e_1, ..., e_k\}$ with $dimV = n = 2k > 7$ (k is an integer) such that, $rank\psi|_{V^\pm} \geq 3$, and $rank\tau|_{V^\pm} = k$. By *Theorem 1* we can assume ψ and τ are simultaneously diagonalizble on the subspaces V^+ and V^- so we may assume $\psi|_{V^+}(f_1, f_1) = \lambda_1$, $\psi|_{V^+}(f_i, f_i) = \lambda_2$ where $i \in \{2, ..., k\}$,$\psi|_{V^-}(e_1, e_1) = \lambda_{k+1}$, $\psi|_{V^-}(e_i, e_i) = \lambda_{k+2}$ $i \in \{k+1, ..., 2k\}$, $\tau|_{V^+}(f_1, f_1) = \eta_1$, $\tau|_{V^+}(f_i, f_i) = \eta_2$ $i \in \{2, ..., k\}$,$\eta|_{V^-}(e_1, e_1) = \eta_{k+1}$, $\tau|_{V^-}(e_i, e_i) = \eta_{k+2}$ where $i \in \{k+1, ..., 2k\}$

.If $\eta_2 \neq \eta_6$ then ψ and τ are simultaneously diagonalizable with respect to φ.

Now we shall consider the case where $k = 4$.

Proof of Theorem 2.1 for $k = 4$:We would to know if $\psi$and τ are simultaneously diagonalize or not.We look at the matrix representing the entries of φ, ψ, and τ :

$$[\phi] \longrightarrow \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} [\psi] \longrightarrow \begin{pmatrix} \lambda_1 & \varpi_7 & \varpi_8 & \varpi_9 & a & b & c & d \\ \varpi_1 & \lambda_2 & \varpi_{10} & \varpi_{11} & e & f & g & h \\ \varpi_2 & \varpi_3 & \lambda_3 & \varpi_{12} & i & j & k & l \\ \varpi_4 & \varpi_5 & \varpi_6 & \lambda_4 & m & n & o & p \\ -a & -e & -i & -m & \lambda_5 & \varpi_{13} & \varpi_{14} & \varpi_{15} \\ -b & -f & -j & -n & \varpi_{19} & \lambda_6 & \varpi_{16} & \varpi_{17} \\ -c & -g & -k & -o & \varpi_{20} & \varpi_{21} & \lambda_7 & \varpi_{18} \\ -d & -h & -l & -p & \varpi_{22} & \varpi_{23} & \varpi_{24} & \lambda_8 \end{pmatrix}$$

$$[\tau] \longrightarrow \begin{pmatrix} \eta_1 & \pi_7 & \pi_8 & \pi_9 & \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ \pi_1 & \eta_2 & \pi_{10} & \pi_{11} & \bar{e} & \bar{f} & \bar{g} & \bar{h} \\ \pi_2 & \pi_3 & \eta_3 & \pi_{12} & \bar{i} & \bar{j} & \bar{k} & \bar{l} \\ \pi_4 & \pi_5 & \pi_6 & \eta_4 & \bar{m} & \bar{n} & \bar{o} & \bar{p} \\ -\bar{a} & -\bar{e} & -\bar{i} & -\bar{m} & \eta_5 & \pi_{13} & \pi_{14} & \pi_{15} \\ -\bar{b} & -\bar{f} & -\bar{j} & -\bar{n} & \pi_{19} & \eta_6 & \pi_{16} & \pi_{17} \\ -\bar{c} & -\bar{g} & -\bar{k} & -\bar{o} & \pi_{20} & \pi_{21} & \eta_7 & \pi_{18} \\ -\bar{d} & -\bar{h} & -\bar{l} & -\bar{p} & \pi_{22} & \pi_{23} & \pi_{24} & \eta_8 \end{pmatrix}$$

Now, recall that we started with a nondegenerate bilinear form φ with basis $\{e_1, e_2, ...e_p, f_1, f_2, ...f_p\}$.Decomposing the vector space V into $V^+ \oplus V^-$ where $W^+ = span\{f_1, f_2, ...f_k\}$ and $W^- = span\{e_1, e_2, ...e_k\}$ we can restrict the linear dependence $R_\varphi = \varepsilon R_\psi + \delta R_\tau\}$ to either V^+ or V^- and the corresponding subspace will obey *Theorem 1.3*. *Remark*: Given the linear dependence $R_\varphi = \varepsilon R_\psi + \delta R_\tau$ with $Rank\psi \geq 3$ and $Rank\tau = dimV$ we can conclude that ψ and τ are simultaneously diagonalize on the subspaces V^+ and V^- using *Theorem 1.3*. Hence the following matrices.

$$[\psi] \longrightarrow \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & a & b & c & d \\ 0 & \lambda_2 & 0 & 0 & e & f & g & h \\ 0 & 0 & \lambda_3 & 0 & i & j & k & l \\ 0 & 0 & 0 & \lambda_4 & m & n & o & p \\ -a & -e & -i & -m & \lambda_5 & 0 & 0 & 0 \\ -b & -f & -j & -n & 0 & \lambda_6 & 0 & 0 \\ -c & -g & -k & -o & 0 & 0 & \lambda_7 & 0 \\ -d & -h & -l & -p & 0 & 0 & 0 & \lambda_8 \end{pmatrix} [\tau] \longrightarrow \begin{pmatrix} \eta_1 & 0 & 0 & 0 & \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ 0 & \eta_2 & 0 & 0 & \bar{e} & \bar{f} & \bar{g} & \bar{h} \\ 0 & 0 & \eta_3 & 0 & \bar{i} & \bar{j} & \bar{k} & \bar{l} \\ 0 & 0 & 0 & \eta_4 & \bar{m} & \bar{n} & \bar{o} & \bar{p} \\ -\bar{a} & -\bar{e} & -\bar{i} & -\bar{m} & \eta_5 & 0 & 0 & 0 \\ -\bar{b} & -\bar{f} & -\bar{j} & -\bar{n} & 0 & \eta_6 & 0 & 0 \\ -\bar{c} & -\bar{g} & -\bar{k} & -\bar{o} & 0 & 0 & \eta_7 & 0 \\ -\bar{d} & -\bar{h} & -\bar{l} & -\bar{p} & 0 & 0 & 0 & \eta_8 \end{pmatrix}$$

We are a step closer to showing that ψ and τ are simultaneously diagonalizable with respect to φ on the whole space V. In [4], it is shown that all but a, b, c, d, $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ are zero. So,

$$[\psi] \longrightarrow \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & a & b & c & d \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ -a & 0 & 0 & 0 & \lambda_5 & 0 & 0 & 0 \\ -b & 0 & 0 & 0 & 0 & \lambda_6 & 0 & 0 \\ -c & 0 & 0 & 0 & 0 & 0 & \lambda_6 & 0 \\ -d & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_6 \end{pmatrix} [\tau] \longrightarrow \begin{pmatrix} \eta_1 & 0 & 0 & 0 & \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ 0 & \eta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta_2 & 0 & 0 & 0 & 0 \\ -\bar{a} & 0 & 0 & 0 & \eta_5 & 0 & 0 & 0 \\ -\bar{b} & 0 & 0 & 0 & 0 & \eta_6 & 0 & 0 \\ -\bar{c} & 0 & 0 & 0 & 0 & 0 & \eta_6 & 0 \\ -\bar{d} & 0 & 0 & 0 & 0 & 0 & 0 & \eta_6 \end{pmatrix}$$

We now prove that if $\eta_2 \neq \eta_6$ then ψ and τ are simultaneously diagonalized by showing that all off diagonal coefficients are zero. Compute $R_\psi(e_1, e_2, e_2, f_1)$ and $R_\psi(e_1, f_2, f_2, f_1)$ gives the following equations:

$$0 = \varepsilon a \lambda_2 + \delta \bar{a} \eta_2, \tag{5}$$

$$0 = \varepsilon a \lambda_6 + \delta \bar{a} \eta_6. \tag{6}$$

Now multiplying equation 5 by λ_6 and equation 6 by λ_2 and then subtracting the resulting equations by each other we arrive at the following,where in Equation 9 we use the eigenvalue relationshipfound in *theorem 1.3*.

$$0 = \delta \bar{a}(\eta_2 \lambda_6 - \eta_6 \lambda_2), \tag{7}$$

$$0 = \delta \bar{a}(\eta_2 \lambda_6^2 - \eta_6 \lambda_2 \lambda_6), \tag{8}$$

$$= \delta \bar{a}(\eta_2 \varepsilon(\delta \eta_6^2 - 1) - \eta_6 \varepsilon(\delta \eta_2 \eta_6 - 1)), \tag{9}$$

$$= \bar(a)(\eta_2 \eta_6^2 - \eta_2 - \eta_2 \eta_6^2 + eta_2) \tag{10}$$

So, since $\eta_2 \neq \eta_6$,

$$0 = \bar{a}(-\eta_2 + \eta_6). \tag{11}$$

$$\bar{a} = 0 \tag{12}$$

Now if we substitute $\bar{a} = 0$ into equation (2.1) we get the following:

$$0 = \varepsilon a\lambda_2 + \delta 0\eta_2 \tag{13}$$

$$0 = \varepsilon a\lambda_2 \tag{14}$$

The latter leads us to the fact that $a = 0$ since $\lambda_2 \neq 0$. By computing $R_\varphi(e_1, e_2, e_2, f_2)$ and $R_\varphi(e_1, f_3, f_3, f_2)$ we get the following.

$$0 = \varepsilon b\lambda_2 + \delta\bar{b}\eta_2 \tag{15}$$

$$0 = \varepsilon b\lambda_6 + \delta\bar{b}\eta_6 \tag{16}$$

These latter equations lead to $b = 0 = \bar{b}$. Similarly we can show that $c = d = \bar{c} = \bar{d} = 0$ by computing the following pairs $(R_\varphi(e_1, e_2, e_2, f_3)$, $R_\varphi(e_1, f_2, f_2, f_3))$ and $(R_\varphi(e_1, e_2, e_2, f_4)$, $R_\varphi(e_1, f_2, f_2, f_4))$and using the same procedure as before.

Note: If $\eta_2 = \eta_6$ then all of the constants $\{a, b, c, d, \bar{a}, \bar{b}, \bar{c}, \bar{d}\}$ cannot be shown to be zero. Our next step is to generalize from the specific case of $dimV = 8$ and $sig(\varphi) = (4, 4)$ to that of $dimV = n$ and $sig(\varphi) = (\frac{n}{2}, \frac{n}{2})$.

4 $Rank\psi = 2$

Given the linear dependence $R_\varphi + \varepsilon R_\psi + \delta R_\tau$ are τ and ψ silmutaneously diagonalizable?

Theorem 3.1:Let V be a vector space with $dimV = 4$, φ is *positive definite*, $Rank\psi = 2$, and $Rank\tau = dimV$. The equation $R_\varphi = \varepsilon R_\psi + \epsilon R_\tau$ has no solution.

Proof of *Theorem 3.1* preamble:It is of interest to find out wether τ and ψ are simultaneously diagonalizable or not. Hence, lets assume that ψ and τ are diagonal to start with and see if $R_\varphi = \varepsilon R_\psi + \epsilon R_\tau$ holds. Now lets plug in the following entries into $R_\varphi(x, y, z, w) = \varepsilon R_\psi(x, y, z, w) + \epsilon R_\tau(x, y, z, w)$.

$$(x, y, z, w) = (e_1, e_2, e_2, e_1), \tag{17}$$

$$(x, y, z, w) = (e_1, e_3, e_3, e_1), \tag{18}$$

$$(x, y, z, w) = (e_1, e_4, e_4, e_1), \tag{19}$$

$$(x, y, z, w) = (e_2, e_3, e_3, e_2), \tag{20}$$
$$(x, y, z, w) = (e_2, e_4, e_4, e_2), \tag{21}$$
$$(x, y, z, w) = (e_3, e_4, e_4, e_3), \tag{22}$$

Which give the following results respectively..

$$1 + \varepsilon\lambda_1\lambda_2 = \delta\eta_1\eta_2 \tag{23}$$

$$1 = \delta\eta_1\eta_3 \tag{24}$$
$$1 = \delta\eta_1\eta_4 \tag{25}$$
$$1 = \delta\eta_2\eta_3 \tag{26}$$
$$1 = \delta\eta_2\eta_4 \tag{27}$$
$$1 = \delta\eta_3\eta_4. \tag{28}$$

By manipulating equation 23 through 28 we conclude that $\eta_1 = \eta_2 = \eta_3 = \eta_4$ which consequently implies that $\delta = 1$. But if $\delta = \eta_1 = ... = \eta_4 = 1$ then $\lambda_1\lambda_2 = 0$. Since $Rank\psi$ was hypothesized to be 2 the latter is a contradiction. It seems that if we begin by assuming ψ and τ are diagonal we arrive at a contradiction. In the next section we begin by only assuming that ψ is diagonal and τ is not.

Proof of theorem 3.1: In this case τ is not assumed to be diagonal and we would like to know what the off diagonal entries are. The following is the matrix representation of τ,

$$\begin{pmatrix} \eta_1 & a & b & c \\ a & \eta_2 & d & e \\ b & d & \eta_3 & f \\ c & e & f & \eta_4 \end{pmatrix}$$

Lets begin by extracting some equations from $R_\varphi(x, y, z, w) = \varepsilon R_\psi(x, y, z, w) + \epsilon R_\tau(x, y, z, w)$.

Plugging in

$$(x, y, z, w) = (e_1, e_2, e_2, e_1), \tag{29}$$
$$(x, y, z, w) = (e_1, e_3, e_3, e_1), \tag{30}$$
$$(x, y, z, w) = (e_1, e_4, e_4, e_1), \tag{31}$$
$$(x, y, z, w) = (e_2, e_3, e_3, e_2), \tag{32}$$

$$(x, y, z, w) = (e_2, e_4, e_4, e_2), \tag{33}$$

$$(x, y, z, w) = (e_3, e_4, e_4, e_3), \tag{34}$$

$$(x, y, z, w) = (e_1, e_3, e_2, e_4), \tag{35}$$

$$(x, y, z, w) = (e_1, e_2, e_3, e_4), \tag{36}$$

$$(x, y, z, w) = (e_4, e_1, e_2, e_3), \tag{37}$$

$$(x, y, z, w) = (e_1, e_3, e_3, e_4), \tag{38}$$

$$(x, y, z, w) = (e_1, e_2, e_2, e_4), \tag{39}$$

$$(x, y, z, w) = (e_2, e_4, e_4, e_3), \tag{40}$$

$$(x, y, z, w) = (e_2, e_1, e_1, e_3), \tag{41}$$

$$(x, y, z, w) = (e_1, e_2, e_2, e_3), \tag{42}$$

produces the following useful equations respectively:.

$$1 + \varepsilon\lambda_1\lambda_2 = \delta\eta_1\eta_2 - \delta a^2, \tag{43}$$

$$1 = \delta\eta_1\eta_3 - \delta b^2, \tag{44}$$

$$1 = \delta\eta_1\eta_4 - \delta c^2, \tag{45}$$

$$1 = \delta\eta_2\eta_3 - \delta d^2, \tag{46}$$

$$1 = \delta\eta_2\eta_4 - \delta e^2, \tag{47}$$

$$1 = \delta\eta_3\eta_4 - \delta f^2, \tag{48}$$

$$0 = \delta(cd - af), \tag{49}$$

$$0 = \delta(cd - be), \tag{50}$$

$$0 = \delta(fa - be), \tag{51}$$

$$0 = \delta(c\eta_3 - bf), \tag{52}$$

$$0 = \delta(c\eta_2 - ae), \tag{53}$$

$$0 = \delta(d\eta_4 - ef), \tag{54}$$

$$0 = \delta(d\eta_1 - ab), \tag{55}$$

$$0 = \delta(\eta_2 b - ad). \tag{56}$$

From Equations 49, 50, and 51 one can see that

$$cd = af = be \tag{57}$$

Its hard to know what a, b, c, d, e, and f should be at this point but with use of the fact that $Rank\psi = 2$ we can narrow it down. We can take an *orthonormal* basis for $Ker\psi : e_3, e_4$ and extend it to an *orthonormal* basis e_1, e_2, e_3, e_4. Let

$$f_1 = e_1 \tag{58}$$

$$f_2 = e_2 \tag{59}$$

$$f_3 = cos\theta e_3 + sin\theta e_4 \tag{60}$$

$$f_4 = -sin\theta e_3 + cos\theta e_4. \tag{61}$$

Therefore

$$[\psi]_f = \psi. \tag{62}$$

Also notice that there exist a θ such that the $(3, 4)$ position is zero in the matrix of τ. Namely, if $f \neq 0$, then with respect to this new basis $f = 0$.

$$cot2\theta = \frac{\eta_3 - \eta_4}{2f} \tag{63}$$

Therefore,

$$[\tau] \to \begin{pmatrix} \eta_1 & \bar{a} & \bar{b} & \bar{c} \\ \bar{a} & \eta_2 & \bar{d} & \bar{e} \\ \bar{b} & \bar{d} & \bar{\eta_3} & 0 \\ \bar{c} & \bar{e} & 0 & \bar{\eta_4} \end{pmatrix}.$$

If the latter is the case then by equation 57

$$\bar{c}\bar{d} = 0 = \bar{b}\bar{e}. \tag{64}$$

One of the cases to consider here is the case

$$\bar{d} = 0 = \bar{e}. \tag{65}$$

If the later is so then

$$\eta_2 = \bar{\eta_4} = \bar{\eta_3} = \pm 1 and \delta = 1 \tag{66}$$

but the latter implies that $b = c = 0$.
Now, due to $R_\varphi(e_1, e_3, e_3, e_2)$ we arrive at

$$0 = \delta(\bar{a}\bar{\eta_3} - \bar{b}\bar{d}) \tag{67}$$

which leads to $\bar{a} = 0$. Finally, due to equation 43 we arrive at the following contradiction.

$$\varepsilon\lambda_1\lambda_2 = 0. \tag{68}$$

The latter uses the fact that $\delta = 1$ and $\eta_1\eta_2 = 1$. Since we originally assumed that $Rank\psi = 2$ the fact that at least one of λ_1 and λ_2 must be zero is a contradiction to the original assumption since then $Rank\psi$ would have to be one or two. In the next section we prove that *theorem 3.1* holds for $n \geq 5$. QED

5 $dimV \geq 5$

Theorem 4.1: Let V be a vector space with $dimV \geq 5$, φ is *positive definite*, $Rank\psi = 2$, and $Rank\tau = dimV$. The set $\{R_\varphi, R_\psi, R_\tau\}$ is never linearly dependent.

Proof:Lets begin by choosing a basis $\{e_i\}$ such that ψ is diagonal with $\psi_{ij} = 0$ for all entries except $ij = 11$ and $ij = 22$ where $\psi_{ij} = \psi(e_i, e_j)$ and φ remains *positive definite*.
If we assume that the set $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent then we have the following relation which results in several cases:

$$c_1 R_\varphi + c_2 R_\psi + c_3 R_\tau = 0. \tag{69}$$

No 2 of c_i can be 0, either. **Case 1**: If only $c_1 = 0$ we have the case where

$$c_2 R_\psi = -c_3 R_\tau \tag{70}$$

which is impossible since $Rank\psi = 2$, $Rank\tau = n$, and $n > 2$.

Case 2: If only $c_2 = 0$ gives us the following:

$$R_\varphi = \frac{-c_3}{c_1} R_\tau \tag{71}$$

But by *theorem 1.3* results we can conclude that τ is a multiple of φ, which has been assumed not to be the case.

Case 3: With $c_3 = 0$ we run into a similar scenario to case 1.

$$c_1 R_\varphi + c_2 R_\psi = 0 \tag{72}$$

Which is impossible since $Rank\psi = 2$, $Rank_\varphi = n$, and $n > 2$.

Case 4: $c_1, c_2,\ c_3$ are all non zero.
It is a result of linear algebra that two operators commute iff they are simultaneously diagonalizable. Notice that if we only consider the subspace $ker\psi$ then it is indeed true that ψ and τ will commute and therefore there must be a change of basis that will make τ diagonal in $ker\psi$.On $ker\psi$ $\tau(e_i, e_j) = \eta_{ij}$ when $i = j$ and zero otherwise. By computing $R_\varphi(e_3, e_j, e_j, e_3) = \varepsilon R_\psi(e_3, e_j, e_j, e_3) + \delta R_\tau(e_3, e_j, e_j, e_3)$ with$j \neq 3, 2, 1$ we arrive at the following.

$$1 = \delta \eta_3 \eta_j, j \neq 3, 2, 1. \tag{73}$$

The latter equations results in $\eta_4 = \eta_5 = ... = \eta_n$.

Next we compute $R_\varphi(e_4, e_j, e_j, e_4) = \varepsilon R_\psi(e_4, e_j, e_j, e_4) + \delta R_\tau(e_4, e_j, e_j, e_4)$with$j \neq 4, 2, 1$ which results in the following.

$$1 = \delta\eta_4\eta_j, j \neq 4, 2, 1 : \tag{74}$$

The latter results in $\eta_3 = \eta_5 = ... = \eta_n$. Together, $\eta_3 = \eta_5 = ... = \eta_n$ and $\eta_4 = \eta_5 = ... = \eta_n$ imply $\eta_3 = \eta_4 = ... = \eta_n$. From 73 we know that $1 = \delta\eta_3\eta_4 = \delta\eta_3^2$. Since η_3^2 is nonnegative, hence $\delta = +1$. We now have the relation $\eta_3 = 1$ which leads to $\eta_3 = \eta_4 = ... = \eta_n = 1$.

Now we will compute $R_\varphi(e_4, e_1, e_3, e_4) = \varepsilon R_\psi(e_4, e_1, e_3, e_4) + \delta R_\tau(e_4, e_1, e_3, e_4)$. The result is

$$0 = \delta(\eta_4 a_{13} - a_{43}a_{14}). \tag{75}$$

But $a_{43} = 0$ since it is an off diagonal element of τ in the $ker\psi$. Consequently from 75 we have $\eta_4 a_{13} = 0$, but since $\eta_4 = 1$ we have $a_{13} = 0$.
Similarly we will show that $a_{23} = 0$ by computing $R_\varphi(e_4, e_2, e_3, e_4) = \varepsilon R_\psi(e_4, e_2, e_3, e_4) + \delta R_\tau(e_4, e_2, e_3, e_4)$. The result is

$$0 = \delta(\eta_4 a_{23} - a_{43}a_{24}). \tag{76}$$

Recall that $a_{43} = 0$.From 76 we have $\eta_4 a_{23} = 0$, but since $\eta_4 = 1$ we have $a_{23} = 0$. The fact that $a_{13} = a_{23} = 0$ plays an important role in the following step.

Lets now compute both $R_\varphi(e_3, e_1, e_1, e_3)$ and $R_\varphi(e_3, e_2, e_2, e_3)$. The results are as follows, respectively:

$$1 = \delta(\eta_1\eta_3 - a_{13}^2), \tag{77}$$

$$1 = \delta(\eta_2\eta_3 - a_{23}^2). \tag{78}$$

But $a_{13} = a_{23} = 0$, $\delta = 1$, and $\eta_3 = 1$. Therefore,

$$1 = \eta_1, \tag{79}$$

$$1 = \eta_2. \tag{80}$$

We now have the important fact that $\eta_1 = \eta_2 = ... = \eta_n = 1$. Before we arrive at our conclusion we need one more result.

Computing $R_\varphi(e_1, e_3, e_3, e_2)$ we arrive at the following:

$$a_{12}\eta_3 = a_{13}a_{32} \tag{81}$$

But $a_{13} = 0$ and $\eta_3 = 1$.
Therefore,

$$a_{12} = 0. \tag{82}$$

But $a_{12} = a_{21}$ since τ is symmetric. Thus, $a_{12} = a_{21} = 0$.

To conclude, we compute $R_\varphi(e_1, e_2, e_2, e_1)$ were $\lambda_1 = \psi(e_1, e_1)$ and $\lambda_2 = \psi(e_2, e_2)$ are the diagonal and only non zero elements of ψ. The result follows:

$$1 = \varepsilon(\lambda_1\lambda_2) + \delta(\eta_1\eta_2 - a_{12}^2) \tag{83}$$

But $\delta = 1$, $a_{12} = 0$, and $\eta_1\eta_2 = 1$. Therefore 85 takes the following form.

$$1 = \varepsilon(\lambda_1\lambda_2) + 1 \tag{84}$$

Which implies that

$$\lambda_1\lambda_2 = 0. \tag{85}$$

But the latter implies that either $\lambda_1 = 0$, $\lambda_2 = 0$, or $\lambda_1 = \lambda_2 = 0$. All of the stated cases lead us to a contradiction because we started out with the assumption that $Rank\psi = 2$ and that consequently implied that λ_1 and λ_2 were nonzero. The case were $n = 4$ was discussed in the prior section and the case were $n = 3$ will be discussed in the next chapter.

6 $dimV = 3$

The case $dimV = 3$ with $Rank\psi = 2$ requires special attention since here $dimKer(\psi) = 1$ and cannot be manipulated in order to argue that one of the of diagonal terms of τ can be zero like we did in $dimV = 4$.

Observation 5.1: If we assume τ and ψ to be simultaneously diagonal then we get the following equations that the eigenvalues of τ and η must satisfy:

$$1 + \varepsilon\lambda_1\lambda_2 = \delta\eta_1\eta_2 \tag{86}$$

$$1 = \delta\eta_1\eta_3 \tag{87}$$

$$1 = \delta\eta_2\eta_3 \tag{88}$$

Now, in the case were we do not assume τ and ψ to be simultaneously diagonalizable we have the following equations relating the eigenvalues of ψ and the entries of τ.

$$1 + \varepsilon\lambda_1\lambda_2 = \delta\eta_1\eta_2 - \delta a^2 \tag{89}$$

$$1 = \delta\eta_1\eta_3 - \delta b^2 \tag{90}$$

$$1 = \delta\eta_2\eta_3 - \delta c^2 \tag{91}$$

$$0 = a\eta_3 - cb \tag{92}$$

$$0 = ab - \eta_1 c \tag{93}$$

$$0 = ac - \eta_2 b. \tag{94}$$

Ex 5.2 The latter set of equations are satisfied if τ and ψ are as follows:
$[\psi] \rightarrow \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{pmatrix} [\tau] \rightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.
The case were $Rank\psi = 3$ with $dimV = 3$ is more complicated with more unknowns. Figuring out a more illuminating way to narrow down the solutions for the unknowns in $Rank\psi \leq 3$ with $dimV = 3$ is an interesting open question.

7 *positive definite* φ, $Rank\psi \geq 3$, $Rank\tau < dimV$

As stated before, it is known that if $\{R_\varphi, R_\psi, R\tau\}$ are linearly dependent then τ and ψ are simultaneously diagonalize under the conditions $dimV \geq 4$,φ is *positive definite*, $Rank\psi \geq 3$,and $Rank\tau = dimV$.If replace the last condition with $\tau < dimV$ maintains the property of ψ and τ being simultaneously diogonalizable. First we need a definition.

Definition 6.1: Let A^g be the generalized inverse of a no invertible operator A, then the following relation shows how to calculate A^g.

$$AA^gA = A \tag{95}$$

Now on to the main result, whose proof is unfinished..

Conjecture 6.2: If $\{R_\varphi, R_\psi, R\tau\}$ are linearly dependent then τ and ψ are simultaneously diagonalize under the conditions $dimV \geq 4$,φ is *positive definitie*, $Rank\psi \geq 3$,$Rank\tau < dimV$.

Method for possible proof: In order to show that τ and ψ are simultaneously diagonalizable we must show that $\tau\psi = \psi\tau$. Lets manipulate the linearly dependent set $\{R_\varphi, R_\psi, R\tau\}$:

$$R_\varphi(\tau^2 x, \tau^2 y, \tau^g \tau z, \tau^g \tau w) = R_\varphi(x, y, \tau z, \tau w) \tag{96}$$

$$R_\psi(\tau^2 x, \tau^2 y, \tau^g \tau z, \tau^g \tau w) = R_{\tau^{g*}\psi\tau}(x, y, \tau z, \tau w) \tag{97}$$

$$R_\tau(\tau^2 x, \tau^2 y, \tau^g \tau z, \tau^g \tau w) = R_\tau(x, y, \tau z, \tau w) \tag{98}$$

Therefore

$$R_\varphi(x, y, \tau z, \tau w) = \varepsilon R_{\tau^{g*}\psi\tau}(x, y, \tau z, \tau w) + \delta R_\tau(x, y, \tau z, \tau w). \tag{99}$$

The latter implies that $R_{\tau^{g*}\psi\tau}(x, y, \tau z, \tau w) = R_\psi(x, y, \tau z, \tau w)$. Which can be rewritten as $R_{\tau^{g*}\psi\tau^2}(x, y, z, w) = R_{\psi\tau}(x, y, z, w)$. Hence,

$$\tau^{g*}\psi\tau^2 = \pm\psi\tau. \tag{100}$$

Equation 99 can be re written as follows:

$$R_\tau(x, y, z, w) - \delta R_{\tau^2}(x, y, z, w) = \varepsilon R_{\tau^{g*}\psi\tau^2}(x, y, z, w) \tag{101}$$

The latter implies that $R_{\tau^{g*}\psi\tau^2}$ is an algebraic curvature tensor and therefore it is a fact that $\tau^{g*}\psi\tau^2 = (\tau^{g*}\psi\tau^2)*$. Finally, by equation 100 and the latter it could potentialy be show that

$$\psi\tau = \tau\psi. \tag{102}$$

The main issues in this proof attempt are found in Equation's 100 and 99. We have relied on inputs of the form $(x, y, \tau z, \tau w)$ rather than the more general (x, y, z, w) and overlooked the sign uncertainty in Equation 100.

8 Conclusion

This document has presented us a theorem that looks at $rank\psi = 2$ under similar conditions as [2] except for the fact that we start of by assuming that ψ is diagonal. It is from [2] that the latter was inspired. In [4] there was the unresolved issue of wether τ and ψ were simultaneously diagonalizable with respect to a non degenerate φ and in this paper we managed to come up with a condition that would enable us to have simultaneous diagnaolization. We ended with a few observations and a conjecture. The observation was the array of equations that characterize the matrix components of ψ and τ in $dimV = 3$ and $Rank\psi = 2$, there was no method i could think of to narrow down a neater set of solutions. Finally, it was conjectured that under an identical hypothesis except to *theorem 1.3* except for $Rank\tau < n$ we would still have simultaneous diagonalization of τ and ψ. The remark and conjecture in the last two sections must still be resolved (also, a general proof for *theorem 2.1* is required).

9 Open Questions

There are a few open questions that i have asked throughout the paper but here are a few more.
1.What happens if ψ and τ are both uninvertible?

2.Are four distinct *canonical curvature tensors* ever linearly dependent, are five? Is there some sort of pattern or behavior that distinguishes an even collection in the set that is assumed to linearly dependent and an odd collection?

3.Will *theorem 3.1* and *theorem 4.1* still hold if ψ was not assume to be diagonal?

10 Acknowledgments

I would like to thank Dr.Dunn and Dr.Trapp for all of their guidance and the opportunity they gave me to work on this REU. I would also like to thank the NSF grant DMS-1156608 and California State University for funding this research.

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