Decoherence

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5/8/2020
Overview.
Closed systems. The Ammonia molecule.
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- Open Quantum Systems.
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Decoherence Models.
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- Photons in a cavity, an experimental study of decoherence.
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Photons in a cavity, an experimental study of decoherence.
Decohere free subspaces.
The Ammonia Molecule, Example of Closed Quantum Systems.
The Ammonia Molecule, an example of a closed quantum system.

Hamiltonian

\[ H \rightarrow \begin{pmatrix} E_0 & -\epsilon \\ -\epsilon & E_0 \end{pmatrix} = \begin{pmatrix} \langle 1|H|1 \rangle & \langle 1|H|2 \rangle \\ \langle 2|H|1 \rangle & \langle 2|H|2 \rangle \end{pmatrix} \]
Hamiltonian

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Let us solve SE for \(|\psi(0)\rangle = |1\rangle\)

- \(i\hbar \partial_t |\psi(t)\rangle = H|\psi(t)\rangle\)
- \(|\psi(t)\rangle = U(t)|\psi(0)\rangle\)
- \(U(t) = e^{\frac{-iHt}{\hbar}}\)
Solution
\[ \psi(t) = e^{-iE/\hbar} \left| \text{one.osf} \right\rangle + i \sin(\epsilon t/\hbar) \left| \text{two.osf} \right\rangle \], superposition principle at work.

State Matrix
\[ \rho(t) = \left| \psi(t) \right\rangle \left\langle \psi(t) \right| \to \cos(\epsilon t/\hbar) - i \cos(\epsilon t/\hbar) \sin(\epsilon t/\hbar) \]

Time evolution of
\[ \text{Tr} \left[ \rho(t) \sigma_z \right] \]
Solution

\[ |\psi(t)\rangle = e^{-iE_0 t/\hbar} (\cos(\epsilon t/\hbar)|1\rangle + i \sin(\epsilon t/\hbar)|2\rangle), \text{ superposition principle at work.} \]
Solution

\[ |\psi(t)\rangle = e^{-iE_0t/t} (\cos(\epsilon t/\hbar)|1\rangle + i \sin(\epsilon t/\hbar)|2\rangle), \text{ superposition principle at work.} \]

State Matrix

\[ \rho(t) = |\psi(t)\rangle\langle\psi(t)| \rightarrow \begin{pmatrix} \cos^2(\epsilon t/\hbar) & -i \cos(\epsilon t/\hbar) \sin(\epsilon t/\hbar) \\ i \sin(\epsilon t/\hbar) \cos(\epsilon t/\hbar) & \sin^2(\epsilon t/\hbar) \end{pmatrix} \]
Solution
\[ |\psi(t)\rangle = e^{-iE_0 t/\hbar} (\cos(\frac{\epsilon t}{\hbar})|1\rangle + i \sin(\frac{\epsilon t}{\hbar})|2\rangle), \text{ superposition principle at work.} \]

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Time evolution of \( Tr[\rho(t)\sigma_z] \)
Coherences

Coherences $\rho_{ij}(t)$, $i \neq j$, are quantum coherences. Their presence is in general a byproduct of the superposition principle.

Closed systems

For the Ammonia molecule, $|\rho_{\text{one.osf/two.osf}}(t)| = \rho_{\text{two.osf/one.osf}}(t) = \cos(\epsilon t\frac{\hbar}{2}) \sin(\epsilon t\frac{\hbar}{2})$. Note the periodic behavior. Closed systems have periodic coherences.

Quantum computation

Quantum coherence is a vital cornerstone to the theory of quantum computation and quantum information. Quantum information is stored within quantum states and the superposition principle is exploited in order to boost computational speed.
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Coherences in closed systems

For the Ammonia molecule, $|\rho_{12}(t)| = \rho_{21}(t) = \cos\left(\frac{\epsilon t}{\hbar}\right) \sin\left(\frac{\epsilon t}{\hbar}\right)$. Note the periodic behavior. Closed systems have periodic coherences.
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Quantum computation

Quantum coherence is a vital cornerstone to the theory of quantum computation and quantum information. Quantum information is stored within quantum states and the superposition principle is exploited in order to boost computational speed.
Open quantum systems
Total system has some Hilbert space $H_S \otimes H_E|\psi_S\rangle \otimes |\psi_E\rangle := |\psi_{SE}\rangle \in H_S \otimes H_E := H_{tot}$.

Dynamics provided by Schrödinger's equation.

$i\hbar \frac{\partial}{\partial t} |\psi_{SE}(t)\rangle = H |\psi_{SE}(t)\rangle$ where $H = H_S + H_E + H_I$, a Hermitian operator in $B(H_{SE})$.

$|\psi_{SE}(t)\rangle = e^{-it\hbar} H$. Just like before.

We can attain the reduced dynamics by partial tracing over the degrees of freedom pertaining to the environment. i.e.

$\rho_S(t) := \text{Tr}_E \{|\psi_{SE}(t)\rangle \langle \psi_{SE}(t)|\}$
Total system has some Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_E$
Open quantum systems

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- $|\psi_{SE}(t)\rangle = e^{-\frac{i}{\hbar}Ht}$. Just like before.
  
  We can attain the reduced dynamics by partial tracing over the degrees of freedom pertaining to the environment. i.e.

  $$\rho_S(t) := Tr_E\{|\psi_{SE}(t)\rangle\langle\psi_{SE}(t)|\}$$
**Definition**

\[ \text{Tr}_E \{ \} : T(\mathcal{H}_S \otimes \mathcal{H}_E) \rightarrow T(\mathcal{H}_S) \]

\[ \text{Tr}_E \{ |\psi_{SE}(t)\rangle\langle\psi_{SE}(t)| \} := \sum_k \langle \phi_R | \psi_{SE}(t) \rangle \langle \psi_{SE}(t) | \phi_R \rangle, \]

where \( \{ |\phi_k \rangle \}_k \) is an ONB for \( \mathcal{H}_E \).
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**Definition**

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where \( \{ |\phi_k\rangle \}_k \) is an ONB for \( \mathcal{H}_E \).

Let us make sure that this map is the correct one.

\[ A_S \to A_S \otimes I_E, \]

\[ \langle A_S \otimes I_E \rangle = \text{Tr}\{ \rho_{SE}(A_S \otimes I_E) \}. \]

But it can be shown that

\[ \text{Tr}\{ \rho_{SE}(A_S \otimes I_E) \} = \text{Tr}\{ \rho_S A_S \} \text{!!!!}. \]
Product state

Product state as an initial state

Assume our initial state to be in a product state.

\[ \rho_{SE}(o) = \rho_S(o) \otimes \rho_E(o) \in D(H_{SE}) := \text{Space of trace class operators over } H_{SE} \text{ with trace one.} \]
Product state as an initial state

Assume our initial state to be in a product state.
\[ \rho_{SE}(0) = \rho_S(0) \otimes \rho_E(0) \in D(\mathcal{H}_{SE}) := \text{Space of trace class operators over } \mathcal{H}_{SE} \text{ with trace one.} \]

Diagonalizing the environmental component

\[ \rho_E(0) = \sum_i p_i |E_i\rangle\langle E_i|. \]
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\[ \rho_E(0) = \sum_i p_i |E_i \rangle \langle E_i| . \]

Non-Unitary Time Evolution

\[ \rho_S(t) = Tr_E\{U(t)\rho_{SE}(0)U^\dagger(t)\} = \sum_{ij} p_i \langle E_j | U(t) | E_i \rangle \rho_S(0) \langle E_i | U^\dagger(t) | E_j \rangle . \]

We short hand this evolution as \( \nu_t \rho_S(0) = \rho_S(t) \). (Dynamical map).
Krauss operators

The operators $\langle E_j | U(t) | E_i \rangle \in \mathcal{B}(\mathcal{H}_S)$ are referred to as krauss operators. These operators characterize the dynamical map seen in the previous slide.
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Definition

A map $\nu_t : D(H_S) \rightarrow D(H_S)$ is said to be a dynamical map if it is a completely positive map, has convex linearity and is trace preserving.
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Complete positivity

$\nu_t \otimes I_n$ required to be positive for all $n$. Without the latter we could end up mapping from positive operators to operators which are not (Negative probabilities).
Convex Linearity, evolving mixed states.

\[ \nu_t \{ \lambda \rho_{S_1}(0) + (1 - \lambda) \rho_{S_2}(0) \} = \lambda \nu_t \rho_{S_1}(0) + (1 - \lambda) \nu_t \rho_{S_2}(0). \]
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Trace preservation.

\[ Tr\{ \nu_t \rho_S(0) \} = 1. \]
The unitary evolution and partial trace approach.

\[ \nu_t \rho_S(0) = \text{Tr}_E \{ U(t) \rho_{SE}(0) U^\dagger \} \]
The unitary evolution and partial trace approach.

\[ \nu_t \rho_S(O) = Tr_E\{U(t)\rho_{SE}(O)U^\dagger\} \]

Partial trace of Von Neumann equation approach

\[ \frac{\partial}{\partial t} \rho_{SE}(t) = -\frac{i}{\hbar}[H_{SE}, \rho_{SE}(t)] \rightarrow \frac{\partial}{\partial t} \rho_S(t) = -\frac{i}{\hbar} Tr_E\{[H_{SE}, \rho_{SE}(t)]\} \]
Reduced Von Neumann equation. Path to Born-Markov master equation.
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Von Neumman: \[ \frac{\partial}{\partial t} \rho_{SE}(t) = -\frac{i}{\hbar} [H_{SE}, \rho_{SE}(t)]. \]
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- Von Neumman: \( \frac{\partial}{\partial t} \rho_{SE}(t) = -\frac{i}{\hbar} [H_{SE}, \rho_{SE}(t)] \).

- Interaction picture: \( \tau(...) := e^{-\frac{i}{\hbar}(H_{S}+H_{E})(...)}e^{\frac{i}{\hbar}(H_{S}+H_{E})} \).
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- Von Neumann in interaction picture (IP):
  $\frac{\partial}{\partial t}\tau(\rho_{SE})(t) = -\frac{i}{\hbar}[\tau(H_{I}), \tau(\rho_{SE})(t)]$
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- Reduced Von Neumann in IP, dropping \( \tau \) for readability:
  \( \frac{\partial}{\partial t} \rho_S(t) = -\frac{i}{\hbar} \text{Tr}_E\{ [H_I(t), \rho_{SE}(t)] \} \)
**Reduced Von Neumann equation. Path to Born-Markov master equation.**

- **Von Neumman:** \( \frac{\partial}{\partial t} \rho_{SE}(t) = -\frac{i}{\hbar} [H_{SE}, \rho_{SE}(t)] \).

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  \[ \frac{\partial}{\partial t} \tau(\rho_{SE})(t) = -\frac{i}{\hbar} [\tau(H_{I}), \tau(\rho_{SE})(t)] \]

- **Reduced Von Neumann in IP, dropping \( \tau \) for readability:**
  \[ \frac{\partial}{\partial t} \rho_{S}(t) = -\frac{i}{\hbar} Tr_{E} \{[H_{I}(t), \rho_{SE}(t)]\} \right) \]

- **Equivalently.**
  \[ \frac{\partial}{\partial t} \rho_{S}(t) = -\frac{i}{\hbar} Tr_{E} \{[H_{I}, \rho_{SE}(0)]\} + \]
  \[ + \frac{i^2}{\hbar^2} \int_{0}^{t} dt_1 Tr_{E} \{[H_{I}(t), [H_{I}(t_1), \rho_{SE}(t_1)]]\} \right) \]
We will work with product state initial conditions.

\[ \rho_{SE}(\text{zero.osf}) = \rho_S(\text{zero.osf}) \otimes \rho_E(\text{zero.osf}) \]

i.e. There are no correlations between the system and the environment. Good approximation for weakly interacting systems.

**Born approximation**

Assuming that the system only weakly affects the bath it is permissible to replace

\[ \rho_S(t) \otimes \rho_E(t) \]

by

\[ \rho_S(t) \otimes \rho_E(\text{zero.osf}) \]

**The Markov approximation**

In order for the reduced Von Neumann equation to be Markovian the integrand must be smooth and sharply peaked in the vicinity of \( t \approx t/\text{one.osf} \). If this holds than we may trade in

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**The Markov approximation**

In order for the reduced Von Neumann equation to be Markovian the integrand must be smooth and sharply peaked in the vicinity of \( t \approx t_1 \). If this holds than we may trade in \( \rho_S(t_1) \otimes \rho_E(0) \) for \( \rho_S(t) \otimes \rho_E(0) \).
Born-Markov master equation

\[
\frac{\partial}{\partial t} \rho_S(t) = -\frac{i}{\hbar} \text{Tr}_E\{[H_I, \rho_S(0) \otimes \rho_E(0)]\} + \\
\frac{i^2}{\hbar^2} \int_{-\infty}^{t} dt_1 \text{Tr}_E\{[H_I(t), [H_I(t_1), \rho_S(t) \otimes \rho_E(0)]]\}
\]
Decoherence Models
Let $E$ be a large, with respect to the system, bosonic bath and $S$ be a two-level system. With a dipole interaction term the dynamics is generated by the Hamiltonian:

$$H_{SE} = \hbar \omega a/\sigma_z + \hbar \sum_k \omega_k b_k^{\dagger} b_k + \hbar \sum_k (g_k b_k^{\dagger} \sigma_+ + g_k b_k \sigma_-).$$

Assuming that the two-level system is in the excited state at $t=0$ we can use the Born-Markov approximation to arrive at the following equation.

$$\frac{\partial}{\partial t} \rho_S(t) = -i \omega a + \Delta \omega a \left[ \sigma_z, \rho_S(t) \right] + \gamma D[\sigma_-] \rho_S(t).$$

$$D[\sigma_-] \rho_S(t) := \sigma_- \rho_S(t) \sigma_+ - \frac{1}{2} \rho_S(t) \sigma_- \rho_S(t) \sigma_+ = \sigma_- \rho \sigma_+ - \frac{1}{2} \rho (\sigma_- \rho \sigma_+) + \frac{1}{2} (\sigma_- \rho \sigma_+ + \sigma_- \rho \sigma_+) - \frac{1}{4} \rho (\sigma_- \rho \sigma_+) - \frac{1}{4} \rho (\sigma_- \rho \sigma_+ + \sigma_- \rho \sigma_+) = -\frac{1}{4} \rho.$$
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$$H_{SE} = \frac{\hbar \omega_a}{2} \sigma_z + \hbar \sum_k \omega_k b_k^\dagger b_k + \hbar \sum_k (g_k b_k + g_k b_k^\dagger)(\sigma_+ + \sigma_-)$$

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$$D[\sigma_-] \rho := \sigma_- \rho \sigma_+ - \frac{1}{2} (\sigma_+ \sigma_- \rho + \rho \sigma_+ \sigma_-)$$
Solving the Master Equation

Solution lives in the Bloch sphere

\[ \rho_S(t) = \frac{1}{2} \left[ l_2 + x(t)\sigma_x + y(t)\sigma_y + z(t)\sigma_z \right], \quad Tr\{\rho_S^2(t)\} \leq 1 \text{ therefore} \]

\[ x^2(t) + y^2(t) + z^2(t) \leq 1. \]
Using the Lindblad Master equation to substitute for \( \frac{\partial}{\partial t} \rho_S(t) \), these equations become

\[
\frac{\partial}{\partial t} z(t) = -\gamma (z(t) + 1) \\
\frac{\partial}{\partial t} y(t) = (\Delta \omega \alpha) x(t) - \gamma y(t) \\
\frac{\partial}{\partial t} x(t) = - (\Delta \omega \alpha) y(t) - \gamma x(t)
\]

with solutions

\[
z(t) = e^{-\gamma t - \gamma t} \\
y(t) = e^{-\gamma t} \sin((\omega \alpha + \Delta \omega \alpha) t) \\
x(t) = e^{-\gamma t} \sin((\omega \alpha + \Delta \omega \alpha) t).
\]
\[
\begin{align*}
\frac{\partial}{\partial t} z(t) &= Tr\{\sigma_z \frac{\partial}{\partial t} \rho_S(t)\} \\
\frac{\partial}{\partial t} y(t) &= Tr\{\sigma_y \frac{\partial}{\partial t} \rho_S(t)\} \\
\frac{\partial}{\partial t} x(t) &= Tr\{\sigma_x \frac{\partial}{\partial t} \rho_S(t)\}
\end{align*}
\]
\[ \frac{\partial}{\partial t} z(t) = \text{Tr}\{\sigma_z \frac{\partial}{\partial t} \rho_S(t)\} \]
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\[ \frac{\partial}{\partial t} y(t) = (\Delta \omega_a)x(t) - \frac{\gamma}{2} y(t) \]
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\[ \frac{\partial}{\partial t} z(t) = Tr\{\sigma_z \frac{\partial}{\partial t} \rho_S(t)\} \]
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\[ \frac{\partial}{\partial t} x(t) = - (\Delta \omega_a) y(t) - \frac{\gamma}{2} x(t) \]

with solutions

\[ z(t) = 2e^{-\gamma t} - 1 \]
\[ y(t) = -e^{-\frac{\gamma t}{2}} \sin((\omega_a + \Delta \omega_a)t) \]
\[ x(t) = e^{-\frac{\gamma t}{2}} \sin((\omega_a + \Delta \omega_a)t) \].
\[
\rho_s(t) \to \begin{bmatrix}
\ e^{-\gamma t} \\
\ e^{-\frac{\gamma t}{2}} \sin((\omega_a + \Delta_a)t) \frac{(1-i)}{2} \\
\ e^{-\frac{\gamma t}{2}} \sin((\omega_a + \Delta_a)t) \frac{(1+i)}{2} \\
\ 1 - e^{-\gamma t}
\end{bmatrix}
\]
GENERAL MASTER EQUATION FOR FINITE DIMENSIONAL HILBERT SPACE $\mathcal{H}_S$

\[
\frac{\partial}{\partial t} \rho_S(t) = \\
-\frac{i}{\hbar} [H'_S, \rho_S(t)] + \sum_{ij}^{N^2} \alpha_{ij}(t) \{ F_i \rho_S(t) F_j^\dagger - \frac{1}{2} F_j^\dagger F_i \rho_S(t) - \frac{1}{2} \rho_S(t) F_j^\dagger F_i \} \\
:= \mathcal{L} \rho_S(t)
\]
GENERAL MASTER EQUATION FOR FINITE DIMENSIONAL
Hilbert SPACE $\mathcal{H}_S$

\[
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\]

:= $\mathcal{L} \rho_S(t)$ The operators $F_i$ are a set of $N^2$ linear operators forming an orthonormal basis for the space $\mathcal{B}(\mathcal{H}_S)$
**General Master Equation for Finite Dimensional Hilbert Space** $\mathcal{H}_S$

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:= \mathcal{L} \rho_S(t)
\]

The operators $F_i$ are a set of $N^2$ linear operators forming an orthonormal basis for the space $\mathcal{B}(\mathcal{H}_S)$

Connecting back to dynamical maps

\[
\frac{\partial}{\partial t} \rho_S(t) = \frac{\partial}{\partial t} \nu_t \rho_S(0) = \frac{\partial}{\partial t} e^{\mathcal{L} t} \rho_S(0) = \mathcal{L} \nu_t \rho_S(0) = \mathcal{L} \rho_S(t)
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Quantum dynamical semigroup

$\mathcal{L}$ is the generator of the dynamical semigroup $\{\nu_t = e^{\mathcal{L} t} | t \geq 0\}$. 
Collisional decoherence, recoilless case.

\[ |x\rangle |E\rangle \xrightarrow{t} |x\rangle |E_x\rangle = |x\rangle S_x |E\rangle. \]

\(S_x\) is the so called S-matrix, a unitary operator. S-matrix simply maps free particle in-states to free particle out-states and excludes information about the interaction.
Evolution of some state $\phi(x) \in L^2(\mathbb{R})$, $|E\rangle \in \mathcal{H}_E$.

$$\{ \int dx \phi(x) |x\rangle \} |E\rangle \xrightarrow{t} \int dx \phi(x) |x\rangle S_x |E\rangle$$
Evolution of some state $\phi(x) \in L^2(\mathbb{R})$, $|E\rangle \in \mathcal{H}_E$.

\[
\{ \int dx \phi(x)|x\rangle \} |E\rangle \xrightarrow{t} \int dx \phi(x)|x\rangle S_x |E\rangle
\]

Reduced Density matrix

\[
\rho_S(x, y) = \phi(x)\phi(y)^* \xrightarrow{t} \rho_S(x, y) \langle E|S_y^\dagger S_x |E\rangle.
\]
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$\{\int dx \phi(x)|x\rangle\}|E\rangle \xrightarrow{t} \int dx \phi(x)|x\rangle S_x|E\rangle$

Reduced Density matrix

$\rho_S(x, y) = \phi(x)\phi(y)^* \xrightarrow{t} \rho_S(x, y)\langle E|S_y^\dagger S_x|E\rangle$.

Scattered photons Long-Wavelength limit

$\langle E|S_y^\dagger S_x|E\rangle \approx e^{-\Lambda t(x-y)^2}$
Collisional Decoherence

\[ \rho_S(x, x') \xrightarrow{t} \rho_S(x, x') e^{-\Lambda t(x-x')^2}. \]

This implies \( \rho_S(x, x', t) = e^{-\Lambda t(x-x')^2} \rho_S(x, x', 0). \)
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The above is a solution to the differential equation

\[ \frac{\partial}{\partial t} \rho_S(x, x', t) = -\Lambda (x - x')^2 \rho_S(x, x', t) \]
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In operator form.

\[ i\hbar \frac{\partial \rho_S(t)}{\partial t} = -i\Lambda [x, [x, \rho_S(t)]] \text{. (Master equation)} \]
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In operator form.

\[ \imath \hbar \frac{\partial \rho_S(t)}{\partial t} = -\imath \Lambda [x, [x, \rho_S(t)]] . \text{(Master equation)} \]

\[ \imath \hbar \frac{\partial \rho_S(t)}{\partial t} = \left[ \frac{p^2}{2m}, \rho_S(t) \right] - \imath \Lambda [x, [x, \rho_S(t)]] . \text{(Including intrinsic dynamics, Master equation)} \]
Different values of $\Lambda$

<table>
<thead>
<tr>
<th>Environment</th>
<th>$\Lambda$ for dust grain, $10^{-3} \text{cm}$</th>
<th>$\Lambda$ for dust particle, $10^{-5} \text{cm}$</th>
</tr>
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<tbody>
<tr>
<td>Cosmic background radiation</td>
<td>$10^6$</td>
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</tr>
<tr>
<td>300k photons</td>
<td>$10^{19}$</td>
<td>$10^{12}$</td>
</tr>
<tr>
<td>Sunlight on earth</td>
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</tr>
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Scattering constant $\Lambda$ and decoherence timescale $\tau_{\Delta x}$.

### Different values of $\Lambda$

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</table>

### Decoherence timescales, $\tau_{\Delta x} := \frac{1}{\Lambda(\Delta x)^2}$

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</tr>
<tr>
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</tr>
<tr>
<td>Best laboratory vacuum</td>
<td>$10^{-14}$</td>
</tr>
<tr>
<td>Air at normal pressure</td>
<td>$10^{-31}$</td>
</tr>
</tbody>
</table>
Superposition of two localized Gaussians, just decoherence.

\[ \phi(x, t = 0) = N_1 e^{-(x-a_1)^2} + N_2 e^{-(x-a_2)^2} \]
Evolution of a Gaussian initial state, decoherence and delocalization.

The probability distribution of our particles position is

\[ P(x, t) := \rho_S(x, x, t). \]
**Spin Chains**

$$ H = -\frac{1}{2} \sum^n_n h_n \sigma^n_z - \frac{1}{2} \sum^{n-1}_n [J^n_x \sigma^n_x \sigma^{n+1}_x + J^n_y \sigma^n_y \sigma^{n+1}_y + J^n_z \sigma^n_z \sigma^{n+1}_z] $$

$$ H \in \mathcal{B}(\mathbb{C}^{\otimes 2N}) $$

Let $N = 10$ and $|\psi(0)\rangle = |1000000000\rangle \in \mathbb{C}^{\otimes 2N}$
Spin Chains Decoherence

- $H = -\frac{1}{2} \sum_n h_n \sigma_z^n - \frac{1}{2} \sum_{n=1}^{N-1} [J_x^n \sigma_x^n \sigma_x^{n+1} + J_y^n \sigma_y^n \sigma_y^{n+1} + J_z^n \sigma_z^n \sigma_z^{n+1}]$

- $H \in \mathcal{B}(C \otimes 2^N)$

- Let $N = 10$ and $|\psi(0)\rangle = \frac{1}{\sqrt{2}} (|1\rangle - |0\rangle)|0000000000\rangle \in C \otimes 2^N$
Decoherence terms damping

For large $N$, the decoherence terms follow an approximate Gaussian dependence $e^{-\Gamma^2 t^2}$. Where $\Gamma$ depends on environmental properties and coupling constants $J_i^n$. 
Hamiltonian

- $H_E = \sum_i (\frac{1}{2m_i} p_i^2 + \frac{1}{2} m_i \omega_i^2 q_i^2)$.
- $H_I = x \otimes \sum_i c_i q_i$.
- $H_S = \frac{1}{2M} p^2 + \frac{1}{2} M \Omega^2 x^2$.

Master equation under Born-Markov approximation

\[
\frac{\partial}{\partial t} \rho_S(t) = -\frac{i}{\hbar} [H_S + \frac{1}{2} M \Delta^2 x^2, \rho_S(t)] - \frac{i\gamma}{\hbar} [x, \{p, \rho_S(t)\}] - D[x[x, \rho_S(t)]] - \frac{f}{\hbar} [x, [p, \rho_S(t)]]
\]

Uncertainty of $x$

$\Delta X^2(t) = \frac{\hbar^2 D}{2m^2 \gamma^2} t$. 
WIGNER TRANSFORM

Wigner transform

\[ W(x, p) := \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{\frac{ipy}{\hbar}} \rho(x + \frac{y}{2}, x - \frac{y}{2}) \, dy \]
Monitoring Coherences with Wigner transform
Decoherence in the lab
Photons states in cavity
Schematic steps

\[
\text{Oven} \quad \rightarrow \quad \text{R} /\text{one.osf} \quad \rightarrow \quad \text{√} /\text{two.osf} \quad (|g\rangle + |e\rangle) /\text{one.osf} \quad \rightarrow \quad \text{√} /\text{two.osf} \quad (|g\rangle + |e\rangle) \quad \text{C} \quad \rightarrow \quad /\text{one.osf} \quad /\text{two.osf} \quad (|g\rangle |\alpha e^{-i\xi}\rangle + |e\rangle |\alpha e^{i\xi}\rangle) \quad /\text{one.osf} \quad /\text{two.osf} \quad (\rightarrow /\text{one.osf} /\text{two.osf} (|\alpha e^{-i\xi}\rangle + |\alpha e^{i\xi}\rangle) |g\rangle + /\text{one.osf} /\text{two.osf} (\rightarrow /\text{one.osf} /\text{two.osf} (|\alpha e^{-i\xi}\rangle + |\alpha e^{i\xi}\rangle) |e\rangle).\]

\[
\text{Detection} \quad \rightarrow \quad |±\rangle = /\text{one.osf} \quad /\text{two.osf} (|\alpha e^{i\xi}\rangle |\alpha e^{-i\xi}\rangle) /\text{three.osf} /\text{two.osf} /\text{four.osf} /\text{nine.osf}.
\]
Schematic steps

- $|\text{Oven}\rangle \xrightarrow{R_1} \frac{1}{\sqrt{2}} (|g\rangle + |e\rangle)$
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- $\frac{1}{\sqrt{2}} (|g\rangle |\alpha e^{-i\xi}\rangle + |e\rangle |\alpha e^{i\xi}\rangle) \xrightarrow{R_2} \frac{1}{2} (|\alpha e^{-i\xi}\rangle + |\alpha e^{i\xi}\rangle) |g\rangle + \frac{1}{2} (-|\alpha e^{-i\xi}\rangle + |\alpha e^{i\xi}\rangle) |e\rangle$. 

Detection $\rightarrow |\pm\rangle = \sqrt{\frac{1}{2} (|\alpha e^{-i\xi}\rangle + |\alpha e^{i\xi}\rangle)}$. 


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  |\pm\rangle = \frac{1}{\sqrt{2}} (|\alpha e^{i\xi}\rangle \pm |\alpha e^{-i\xi}\rangle)$
Photon field

State of photonic field left behind in cavity.

\[ |\pm\rangle = \frac{1}{\sqrt{2}} (|\alpha e^{i\xi}\rangle \pm |\alpha e^{-i\xi}\rangle) \]

If the atom is detected to be in the state \( |g\rangle \) the field is in the field \( |+\rangle \), if the atom is detected in the state \( |e\rangle \) the field is in the state \( |-\rangle \).
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Mesoscopic distinguishability.

To measure the degree to which components \(|\alpha e^{i\xi}\rangle\) and \(|\alpha e^{-i\xi}\rangle\) represent mesoscopically or macroscopically distinguishable states- we consider \(|\langle \alpha e^{i\xi} | \alpha e^{-i\xi} \rangle|^2 = e^{-4|\alpha|^2 \sin^2 \xi}\).
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\[ |\langle e^{i\xi} | \alpha e^{-i\xi}\rangle|^2 = e^{-4|\alpha|^2 \sin^2 \xi} \]
For mean number of photons \(|\alpha|^2 \approx 10\) and \(\xi = 0.31\)

\[ |\langle e^{i\xi} | \alpha e^{-i\xi}\rangle|^2 < 3 \times 10^{-5}. \]
Decoherence

State $|\pm\rangle = \frac{1}{\sqrt{2}} (|\alpha e^{i\xi}\rangle \pm |\alpha e^{-i\xi}\rangle)$ are superpositions in the observable basis.
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- Decoherence is expected.

- To measure the time dependence of decoherence a second rubidium atom is sent through at varying wait time.

- It can be shown that under zero decoherence $P_{ee} = 1$, probability of first and second atom being detected in the excited state. On the other hand, under full decoherence $P_{eg} = 1$, the probability of finding the second atom in the ground state is 1.

- A useful measuring tool of decoherence.

\[ \eta(\tau) = P_{ee}(\tau) - P_{eg}(\tau). \]
Two-Atom Correlation Signal. \( \eta(\tau) := P_{ee}(\tau) - P_{eg}(\tau) \).
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Decoherence time scale \( T_d = \frac{T_r}{2|\alpha|^2 \sin^2 \xi}. \)
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Decoherence time scale \( T_d = \frac{T_r}{2|\alpha|^2 \sin^2 \xi} \). \( T_r \) damping time of cavity.
Decoherence Free Subspaces
Total system has some *Hilbert* space $\mathcal{H}_S \otimes \mathcal{H}_E$.
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$|\psi_S\rangle \otimes |\psi_E\rangle := |\psi_{SE}\rangle \in \mathcal{H}_S \otimes \mathcal{H}_E := \mathcal{H}_{\text{tot}}.$
Total system has some Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_E$

$$|\psi_S\rangle \otimes |\psi_E\rangle := |\psi_{SE}\rangle \in \mathcal{H}_S \otimes \mathcal{H}_E := \mathcal{H}_{\text{tot}}.$$

Dynamics provided by Schrödinger’s equation.

$$i\hbar \partial_t |\psi_{SE}(t)\rangle = H|\psi_{SE}(t)\rangle$$ where $H = H_S + H_E + H_I$, a Hermitian operator in $\mathcal{B}(\mathcal{H}_{SE})$.

$$|\psi_{SE}(t)\rangle = e^{-\frac{it}{\hbar}H}. \text{ Just like before.}$$
Let $H_I = \sum_i S_i \otimes E_i$. 
Decoherence in Measurement Limit

- Let $H_l = \sum_i S_i \otimes E_i$.
- Time evolution of product state
  $$\rho_{SE}(0) = \{\sum_{1,m} c_m c_m^* |s_l\rangle\langle s_m| \} \otimes |E_o\rangle\langle E_o|.$$
Let $H_l = \sum_i S_i \otimes E_i$.

Time evolution of product state
\[
\rho_{SE}(t) = \{\sum_{1,m} c_m c_m^* |s_l\rangle\langle s_m| \} \otimes |E_o\rangle\langle E_o|.
\]

\[
\rho_S(t) = \quad Tr_E \{ e^{\frac{-it}{\hbar}} \sum_k S_k \otimes E_k ( \{ \sum_{1,m} c_m c_m^* |s_l\rangle\langle s_m| \} \otimes |E_o\rangle\langle E_o| ) e^{\frac{it}{\hbar}} \sum_k S_k \otimes E_k \}
\]

This partial trace in general reduces to some state of the form,
\[
\rho_S(t) = \sum_{l,m} a_l(t) a_m^*(t) |s_l\rangle\langle s_m|
\]

with $a_l(t)a_m^*(t) \rightarrow 0$ as $t \rightarrow \infty$ for $l \neq m$. 
Decoherence Free Subspaces.

From what space $\mathcal{H}_C \subset \mathcal{H}_A$ may we construct superpositions $\sum_i c_i |s_i\rangle$ that are immune to decoherence?
Decoherence Free Subspaces.

From what space \( \mathcal{H}_C \subset \mathcal{H}_A \) may we construct superpositions \( \sum_l c_l |s_l\rangle \) that are immune to decoherence? i.e.

\[
Tr_E\{ e^{-\frac{i}{\hbar} \sum_k S_k \otimes E_k (\sum_{l,m} c_l c_m^* |s_l\rangle \langle s_m| \otimes |E_0\rangle \langle E_0|) e^{\frac{i}{\hbar} \sum_k S_k \otimes E_k} } =
\]

\[
= \sum_{l,m} c_l c_m^* |s_l\rangle \langle s_m|
\]
Decoherence Free Subspaces.

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$$= \sum_{l,m} c_l c^*_m \ket{s_l} \bra{s_m}$$

Need $\{\phi_i\}_i$ ONB, with the exotic property of forming a degenerate eigen space for all $S_k$. 
Decoherence Free Subspaces.

From what space $\mathcal{H}_C \subset \mathcal{H}_A$ may we construct superpositions $\sum_l c_l |s_l\rangle$ that are immune to decoherence? i.e.

$$\text{Tr}_E\{e^{-it\frac{\hbar}{\lambda}} \sum_k S_k \otimes E_k \left( \sum_{l,m} c_l c_m^* |s_l\rangle \langle s_m| \otimes |E_0\rangle \langle E_0| \right) e^{it\frac{\hbar}{\lambda}} \sum_k S_k \otimes E_k \} =$$

$$= \sum_{l,m} c_l c_m^* |s_l\rangle \langle s_m|$$

Need $\{\phi_i\}_i$ ONB, with the exotic property of forming a degenerate eigen space for all $S_k$.

$$|\psi_{SE}(t)\rangle = e^{-it\frac{\hbar}{\lambda}} \sum_k S_k \otimes E_k \sum_l c_l |s_l\rangle \otimes |E_0\rangle =$$

$$= \sum_l c_l e^{-it\frac{\hbar}{\lambda}} \sum_k \lambda_k l_A \otimes E_k |s_l\rangle \otimes |E_0\rangle = \sum_l c_l |s_l\rangle \otimes \left[ e^{-it\frac{\hbar}{\lambda}} \sum_k \lambda_k E_k |E_0\rangle \right]$$
Let us now partial trace the corresponding density matrix.

\[ \rho_S(t) = \sum_{l,m} c_l c_m^* |s_l\rangle\langle s_m| \text{Tr}_E\left[ e^{-\frac{it}{\hbar}} \sum_k \lambda_k E_k |E_0\rangle\langle E_0| e^{\frac{it}{\hbar}} \sum_k \lambda_k E_k \right] \]
Let us now partial trace the corresponding density matrix.

\[ \rho_S(t) = \sum_{l,m} c_l c^*_m |s_l\rangle \langle s_m| \text{Tr}_E [e^{-i t/\hbar} \sum_k \lambda_k E_k |E_0\rangle \langle E_0| e^{i t/\hbar} \sum_k \lambda_k E_k ] \]

The trace term is just one since density matrices have trace one under unitary evolution.

\[ \rho_S(t) = \sum_{l,m} c_l c^*_m \]

- Decoherence free!!
Consider a system of $N$ qubits coupled to its environment in the following way.

$|0\rangle_j \rightarrow |0\rangle_j$

$|1\rangle_j \rightarrow e^{i\phi} |1\rangle_j$.

$j$ indexes over all qubits.
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$$|1\rangle_j \rightarrow e^{i\phi}|1\rangle_j.$$ 

$j$ indexes over all qubits. Let the initial state be

$$|\psi\rangle_0 = \bigotimes_{j=1}^{N}(a_j|0\rangle_j + b_j|1\rangle_j).$$
Consider a system of $N$ qubits coupled to its environment in the following way.

$$|0\rangle_j \rightarrow |0\rangle_j$$

$$|1\rangle_j \rightarrow e^{i\phi}|1\rangle_j.$$ 

$j$ indexes over all qubits. Let the initial state be

$$|\psi\rangle_0 = \bigotimes_{j=1}^{N} (a_j|0\rangle_j + b_j|1\rangle_j).$$

The dephasing process evolves our system into the following state.

$$|\psi\rangle_\phi = \bigotimes_{j=1}^{N} (a_j|0\rangle_j + b_j e^{i\phi}|1\rangle_j)$$

with a probability $p_\phi$.
The ensemble $\{|\psi\rangle_\phi, p_\phi\}$ can be expressed equivalently as a mixed state.

$$\rho = \int p_\phi |\psi\rangle_\phi \langle \psi | d\phi$$
The ensemble \( \{ |\psi\rangle_\phi, p_\phi \} \) can be expressed equivalently as a mixed state.

\[ \rho = \int p_\phi |\psi\rangle_\phi \langle \psi | d\phi \]

\[ |\psi\rangle_\phi \langle \psi | \rightarrow \bigotimes_{j=1}^{N} \left[ \begin{array}{cc} |a_j|^2 & a_j b_j^* e^{-i\phi} \\ a_j^* b_j e^{i\phi} & |b|^2 \end{array} \right]. \]
The ensemble $\{ |\psi\rangle_\phi, p_\phi \}$ can be expressed equivalently as a mixed state.

$$
\rho = \int p_\phi |\psi\rangle_\phi \langle \psi | d\phi
$$

$$
|\psi\rangle_\phi \langle \psi | \rightarrow \bigotimes_{j=1}^N \begin{bmatrix}
|a_j|^2 & a_j b_j^* e^{-i\phi} \\
\bar{a_j} b_j e^{i\phi} & |b|^2
\end{bmatrix}.
$$

For a Gaussian distribution $p_\phi = \left(4\pi\alpha^{-1/2}\right)e^{-\frac{\phi^2}{4\alpha}}$ we have

$$
\rho \rightarrow \bigotimes_{i=1}^N \begin{bmatrix}
|a_j|^2 & a_j b_j^* e^{-\alpha} \\
\bar{a_j} b_j e^{-\alpha} & |b|^2
\end{bmatrix}.
$$

There is indeed decoherence present, let's look for some DFS.
For starters let’s consider the case $N = 2$. The dephasing for each of the constituents of the corresponding Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$ is summarized by the following.

- $|00\rangle \rightarrow |00\rangle$
- $|01\rangle \rightarrow e^{i\phi}|01\rangle$
- $|10\rangle \rightarrow e^{i\phi}|10\rangle$
- $|11\rangle \rightarrow e^{2i\phi}|11\rangle$. 
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$\text{Span}\{|01\rangle, |10\rangle\}$?
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$$\text{Span}\{|01\rangle, |10\rangle\}?$$

check...

$$|\psi\rangle = a|01\rangle + b|10\rangle \rightarrow ae^{i\phi}|01\rangle + be^{i\phi}|10\rangle = e^{i\phi}|\psi\rangle$$

It works!!
For $N = 3$ the largest DFS is $Span\{|001\rangle, |010\rangle, |100\rangle\}$ or $Span\{|011\rangle, |101\rangle, |110\rangle\}$
For $N = 3$ the largest DFS is $Span\{|001\rangle, |010\rangle, |100\rangle\}$ or $Span\{|011\rangle, |101\rangle, |110\rangle\}$ In general $\max[\dim(DFS)] = \left(\frac{N}{F_N^2}\right)$
For $N = 3$ the largest DFS is $\text{Span}\{ |001\rangle, |010\rangle, |100\rangle \}$ or $\text{Span}\{ |011\rangle, |101\rangle, |110\rangle \}$. In general $\max[\text{dim}(\text{DFS})] = \binom{\frac{N}{2}}{F_\frac{N}{2}}$. A textbook application of Stirling’s formula yields the following.

$$\frac{|\max[\text{Dim}(\text{DFS})] - 2^N|}{2^N} \to 1.$$
For $N = 3$ the largest DFS is $Span\{ |001\rangle, |010\rangle, |100\rangle \}$ or $Span\{ |011\rangle, |101\rangle, |110\rangle \}$ In general $\max[\dim(DFS)] = \left( F\left( \frac{N}{2} \right) \right)$ A textbook application of stirling’s formula yields the following.

$$\frac{|\max[Dim(DFS)] - 2^N|}{2^N} \rightarrow 1.$$ 

The dimension of the optimal DFS becomes relatively close to the dimension of the system for large $N$. 
Study robustness of DFS under perturbations. Find a way to simulate large spin environments. Decoherence theory in infinite dimensional Hilbert spaces and extending SBS theory to such systems.
Future work

- Study robustness of DFS under perturbations.
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Study robustness of DFS under perturbations.
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Decoherence theory in infinite dimensional Hilbert spaces and extending SBS theory to such systems.
Thank you for your time.


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