Radio Labeling
Summer Research Final Report

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Introduction

Given a graph G with vertex set $V(G) = \{v_1, v_2, v_3, ..., v_n\}$. The *distance* denoted, $d(v_i, v_j)$, is the length of the shortest path between the two vertices v_i and v_j. The *diameter*, denoted by $diam(G)$, is the greatest distance between any two vertices in a graph. A *radio-labeling* is a function $f : V(G) \longrightarrow \{0, 1, 2, 3, ...\}$ satisfying the condition:

$$\mid f(v_i) - f(v_j) \mid \geq diam(G) - d(v_i, v_j) + 1 \tag{1}$$

Each vertex can be considered a radio station in the vicinity of a city G and we assign each station a channel number that satisfies the inequality (1). Therefore, $f(v_i)$ is sometimes referred to as the channel for v_i. In addition, the *span* of f is defined as $max_{u,v \in V(G)}\ \{\mid f(u) - f(v) \mid\}$. In other words, it is the difference between the highest and the lowest channel assignment. The *radio number* of a graph G, denoted by $rn(G)$, is defined as the minimum span of all possible radio labeling of the graph.

Finding the pattern for P_n when n is even

We want to develop a pattern to determine the order in which we assign channels to the vertices in a path graph P_n, so that we can obtain a radio labeling for P_n.

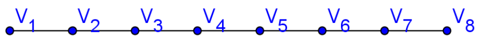

Figure 1: A labeling of the vertices for P_8.

Let V(P_n)= $\{$ v_1, v_2, v_3, v_4,..., v_n $\}$ oriented as in the figure above, so that we number the vertices form left to right. Now we are going to rewrite $V(P_n) = \{x_i \mid\ i \in \{1, 2, 3, ..., n\}\}$, so that i will represent the order in which we assign the channel number to the given vertex. Therefore, for our labeling $f : V(G) \longrightarrow \{0, 1, 2, ...\}$ we have the following: $\forall i \in \mathbb{N}$, $f(x_i) < f(x_{i+1})$. In addition, the lowest channel number will always be 0 in order to find the minimum span, by just looking at $f(x_n)$, and we will always add to the channel number the lowest possible amount (*i.e.*, $f(x_{i+1}) = f(x_i) + diam(G) - d(x_i, x_j) + 1$) to satisfy the inequality, (1), for radio-labeling. For the even path graphs we start with $f(x_1) = 0$. Thus, for all even paths, we found a pattern. First, we start with the center vertex, $v_{\frac{n}{2}}$ (*i.e.*, let $x_1 = v_{\frac{n}{2}}$). Then we move to the right most vertex in the path, v_n (*i.e.*, let $x_2 = v_n$). Next we label the vertex right before the middle vertex ($v_{\frac{n}{2}}$), $v_{\frac{n}{2}-1}$. Afterwards, we move to the vertex right before the right most vertex, v_{n-1}. The order continues as the following: $v_{\frac{n}{2}-2}$, v_{n-2}, $v_{\frac{n}{2}-3}$, v_{n-3},..., until all the vertices are labeled.

So we came up with the formulas:

$$x_i = \begin{cases} v_{\frac{n}{2}-k}, & \text{if } i = 2k+1 \\ v_{n-(k-1)}, & \text{if } i = 2k \end{cases}$$
$$= \begin{cases} v_{\frac{n}{2}-\frac{i-1}{2}}, & \text{if } i \text{ is odd} \\ v_{n-(\frac{i}{2}-1)}, & \text{if } i \text{ is even} \end{cases}$$
$$= \begin{cases} v_{\frac{n+1}{2}-\frac{i}{2}}, & \text{if } i \text{ is odd} \\ v_{n+1-\frac{i}{2}}, & \text{if } i \text{ is even} \end{cases}$$

When assigning the channel we always start with $f(x_1) = 0$ and we add the smallest amount possible to satisfy the inequality (1). We came up with the following formulas to find the channel number and the span of our labeling f:

$$f(x_i) = \begin{cases} (\frac{n(i-1)}{2}) - (\frac{n-1}{2}), & \text{if } i \text{ is odd} \\ (\frac{n(i-1)}{2}) - (\frac{n}{2} - 1), & \text{if } i \text{ is even} \end{cases}$$
$$= \begin{cases} (\frac{n}{2}i) - n + (\frac{1}{2}), & \text{if } i \text{ is odd} \\ (\frac{n}{2}i) - n + 1, & \text{if } i \text{ is even} \end{cases}$$

Therefore, the span is:

$$f(x_n) = \frac{n}{2}(n) - n + 1$$
$$= \frac{n^2}{2} - n + 1$$

Figures 2-6 are examples of the pattern we found for even paths $P_2, P_4, ..., P_{10}$.

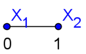

Figure 2: A radio-labeling of P_2

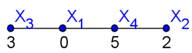

Figure 3: A radio-labeling of P_4

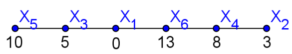

Figure 4: A radio-labeling of P_6

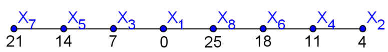

Figure 5: A radio-labeling of P_8

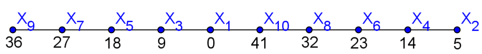

Figure 6: A radio-labeling of P_{10}

Proof that the pattern for even paths is a radio-labeling

Now we will need to show that the labeling pattern we found for even paths is a radio-labeling.
Recall that the pattern we found is:

$$f(x_i) = \begin{cases} (\frac{n}{2}i) - n + (\frac{1}{2}), & \text{if } i \text{ is odd} \\ (\frac{n}{2}i) - n + 1, & \text{if } i \text{ is even} \end{cases}$$

where

$$= \begin{cases} v_{\frac{n+1}{2}-\frac{i}{2}}, & \text{if } i \text{ is odd} \\ v_{n+1-\frac{i}{2}}, & \text{if } i \text{ is even} \end{cases}$$

Using these formulas we can prove that the pattern is in-fact a radio labeling

$< Proof >$
NTS: $\forall i, j \in \{1, 2, 3, ..., n\},\ i \neq j,\ \mid f(x_i) - f(x_j) \mid \geq diam(P_n) - d(x_i, x_j) + 1$

Note 1: $diam(P_n) - d(x_i, x_j) + 1 = (n-1) - d(x_i, x_j) + 1 = n - d(x_i, x_j)$
Since $\forall\ n \in \mathbb{N}$, $diam(P_n) = n - 1$

<u>*Case* 1</u>: i and j are both even
$\Rightarrow \exists k, q \in \mathbb{N} \ni i = 2k$ and $j = 2q,\ k, q \in \mathbb{N}$
$\mid f(x_i) - f(x_j) \mid = \mid [\frac{(i-1)n}{2} - \frac{n-2}{2}] - [\frac{(j-1)n}{2} - \frac{n-2}{2}] \mid$
$= \mid \frac{in-n-n+2}{2} + \frac{-jn+n+n-2}{2} \mid$
$= \mid \frac{in-jn}{2} \mid = \mid \frac{(i-j)n}{2} \mid \geq n - d(x_i, x_j)$
because $\mid i - j \mid > 1 \Rightarrow \mid i - j \mid \geq 2$ (*since* i, j are both even)
$\Rightarrow \mid \frac{n(i-j)}{2} \mid = \frac{n}{2} \mid i - j \mid \geq (\frac{n}{2})(2) = n > n - 1 \geq n - d(x_i, x_j)$
$(since\ d(x_i, x_j) \geq 1 \Rightarrow -d(x_i, x_j) \leq -1 \Rightarrow n - d(x_i, x_j) \leq n - 1)$
$\mid f(x_i) - f(x_j) \mid = \mid \frac{n(i-j)}{2} \mid \geq n - d(x_i, x_j)$

<u>*Case* 2</u>: i and j are both odd
$\Rightarrow \exists k, q \in \mathbb{N}$ s.t. $i = 2k + 1$ and $j = 2q + 1$
$\mid f(x_i) - f(x_j) \mid = \mid [\frac{(i-1)n}{2} - \frac{n-1}{2}] - [\frac{(j-1)n}{2} - \frac{n-1}{2}] \mid$
$= \mid \frac{in-n-n+1}{2} + \frac{-jn+n+n-1}{2} \mid$
$= \mid \frac{in-jn}{2} \mid = \mid \frac{(i-j)n}{2} \mid \geq n - d(x_i, x_j)$
$\Rightarrow \mid i - j \mid \geq 2$(*since* i, j are both odd)
$\Rightarrow \mid \frac{n(i-j)}{2} \mid = \frac{n}{2} \mid i - j \mid \geq (\frac{n}{2})(2) = n > n - 1 \geq n - d(x_i, x_j)$
$(since\ d(x_i, x_j) \geq 1 \Rightarrow -d(x_i, x_j) \leq -1 \rightarrow n - d(x_i, x_j) \leq n - 1)$
$\mid f(x_i) - f(x_j) \mid = \mid \frac{n(i-j)}{2} \mid \geq n - d(x_i, x_j)$

Case 3: i and j are of different parity
Without loss of generality say i is even and j is odd.

Note 2: $i = 2k \Rightarrow x_i = v_{n-(k-1)}$ and $j = 2q+1 \Rightarrow x_j = v_{\frac{n}{2}-q}$
$\Rightarrow n - d(x_i, x_j) = n - d(v_{n-(\frac{i}{2})+1}, v_{\frac{n-j+1}{2}}) = n - ((n - (\frac{i}{2}) + 1) - (\frac{n-j+1}{2}))$
$= n - ([n-(k-1)] - [\frac{n}{2} - q]) = n - n + k - 1 + \frac{n}{2} - q = \frac{n}{2} + k - q - 1$

$| f(x_i) - f(x_j) | = | [\frac{(i-1)n}{2} - \frac{n-2}{2}] - [\frac{(j-1)n}{2} - \frac{n-1}{2}] |$
$= | \frac{in-n-n+2}{2} + \frac{-jn+n+n-1}{2} |$
$= | \frac{in-jn+1}{2} | = | \frac{(i-j)n+1}{2} | \geq (i-j)$

Case 3.1: $i > j$
$\Rightarrow | i-j | = i-j \geq 1$

Case 3.1.1: $i - j = 1$
$| \frac{n(1)+1}{2} | \geq \frac{n}{2} = n - d(x_i, x_j)$

$(Since\ \frac{n}{2} + k - q - 1 = \frac{n}{2} + 1 + q - q - 1 = \frac{n}{2}\ (Since\ i - j = 1$
$\Rightarrow 2k - (2q+1) = 1 \Rightarrow 2k - 2q - 1 = 1 \Rightarrow 2k - 2q = 2 \Rightarrow k - q = 1 \Rightarrow k = 1 + q))$
$\Rightarrow | \frac{n+1}{2} | \geq n - d(x_i, x_j)$

Case 3.1.2: $i - j \neq 1$
$\Rightarrow i - j > 1 \Rightarrow | i - j | \geq 2$
$\Rightarrow | \frac{n(i-j)+1}{2} | \geq | \frac{n(2)+1}{2} | = | \frac{2n+1}{2} | = | n + \frac{1}{2} | > n - 1 \geq n - d(x_i, x_j)$

Case 3.2: $i < j$
$\Rightarrow i - j < 0 \Rightarrow | i - j | \geq 1$

Case 3.2.1: $i - j = -1$
$| \frac{n(-1)+1}{2} | = | \frac{1-n}{2} | = | \frac{n-1}{2} | \geq \frac{n-2}{2} = n - d(x_i, x_j)$

(Since $\frac{n}{2} + k - q - 1 = \frac{n}{2} + k - k - 1 = \frac{n-2}{2}$ $(Since\ i - j = -1$
$\Rightarrow 2k - (2q+1) = -1 \Rightarrow 2k - 2q - 1 = -1 \Rightarrow 2k - 2q = 0 \Rightarrow k - q = 0 \Rightarrow k = q))$
$\Rightarrow | \frac{n-1}{2} | \geq n - d(x_i, x_j)$

Case 3.2.2 $i - j \neq -1$
$| i - j | \geq 2 \Rightarrow i - j \leq -2 \Rightarrow j - i \geq 2$
$| \frac{n(i-j)+1}{2} | = | \frac{-n(j-i)+1}{2} | = | \frac{n(j-i)-1}{2} | \geq | \frac{n(2)-1}{2} | = | n - \frac{1}{2} |$
$= n - \frac{1}{2} \geq n - 1 \geq n - d(x_i, x_j) \Rightarrow | n - \frac{1}{2} | \geq n - d(x_i, x_j)$

Finding the pattern for P_n when n is odd

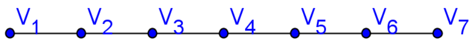

Figure 7: A labeling of the vertices for P_7.

Let V(P_n)= { v_1, v_2, v_3, v_4,..., v_n } oriented as in the figure above, so that we number the vertices from left to right. Now we are going to rewrite $V(G) = \{x_i \mid i \in \{1, 2, ..., n\}\}$, so that i will represent the order in which we assign the channel number to the given vertex. Therefore, for our labeling $f : V(G) \longrightarrow \{0, 1, 2, ...\}$ we have the following: $\forall i \in \mathbb{N}\ f(x_i) < f(x_{i+1})$. In addition the lowest channel number will always be 0 and we will always add to the channel number to lowest possible amount (*i.e.*, $f(x_{i+1}) = f(x_i) + diam(G) - d(x_i, x_j) + 1$) to satisfy the inequality (1), for radio labeling. For the path graph of odd path, we start with $f(x_1) = 0$. Thus, we found a pattern for all odd paths. First, we start with the center vertex, $v_{\frac{n+1}{2}}$ (*i.e.*, let $x_1 = v_{\frac{n+1}{2}}$). Then we move to the right most vertex in the path, v_n (*i.e.*, let $x_2 = v_n$). Next, we label the first vertex v_1. Afterwards, we move to the vertex after the center one, $v_{\frac{n+1}{2}+1}$. Then the order continues as follows: v_2, $v_{\frac{n+1}{2}+2}$, v_3, $v_{\frac{n+1}{2}+3}$,..., until all the vertices are labeled. We came up with the formulas:

$$x_1 = v_{\frac{n+1}{2}}$$
$$x_2 = v_n$$
$$\text{for } i > 2 :$$

$$\begin{aligned} x_i &= \begin{cases} v_k, & \text{if } i = 2k+1 \\ v_{\frac{n+1}{2}+(k-1)}, & \text{if } i = 2k \end{cases} \\ &= \begin{cases} v_{\frac{i-1}{2}}, & \text{if } i \text{ is odd} \\ v_{\frac{n+1}{2}+(\frac{i}{2}-1)}, & \text{if } i \text{ is even} \end{cases} \\ &= \begin{cases} v_{\frac{i-1}{2}}, & \text{if } i \text{ is odd} \\ v_{\frac{n+i-1}{2}}, & \text{if } i \text{ is even} \end{cases} \end{aligned}$$

Now we noticed when assigning the channel numbers we had to jump $f(x_4)$ by one every time (*Since* $\mid f(x_2)$- $f(x_4) \mid\ \geq diam(P_n)$ - $d(x_2, x_4) + 1 \Rightarrow \mid \frac{n+1}{2} - (n+1) \mid = \mid \frac{-n-1}{2} \mid = \mid \frac{n+1}{2} \mid \geq \frac{n+3}{2}$ thus we arrive at a contradiction and we must add 1 to the channel number to satisfy the inequality).

Because of this, we have the following formulas for finding the channel numbers for odd paths P_n and the span of our labeling:

$$f(x_1) = 0$$
$$f(x_2) = \frac{n+1}{2}$$
$$f(x_3) = \frac{n+3}{2}$$
$$f(x_4) = n+1+1$$

for $i > 4$,

$$f(x_i) = \begin{cases} (\frac{n(i-2)+5}{2}), & \text{if } i \text{ is odd} \\ (\frac{n(i-2)+4}{2}), & \text{if } i \text{ is even} \end{cases}$$

Therefore, the span is:

$$\begin{aligned} f(x_n) &= \frac{n(n-2)+5}{2} (\text{ since } n \text{ is odd.}) \\ &= \frac{n^2-2n+5}{2} \\ &= \frac{(n-1)^2}{2} + 2 \end{aligned}$$

Figures 8-11 are examples of the pattern we found for odd paths $P_3, P_5, ..., P_9$.

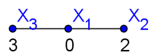

Figure 8: A radio-labeling of P_3

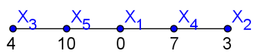

Figure 9: A radio-labeling of P_5

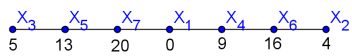

Figure 10: A radio-labeling of P_7

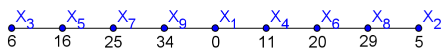

Figure 11: A radio-labeling of P_9

Proof That the Pattern for an Odd Path is a Radio-Labeling

Now we need to show that the labeling pattern we found for odd paths is a radio labeling. The formulas we found are as follows:

$$f(x_i) = \begin{cases} 0 & \text{if } i = 1 \\ \frac{n+1}{2} & \text{if } i = 2 \\ \frac{n+3}{2} & \text{if } i = 3 \\ n+2 & \text{if } i = 4 \\ \frac{n(i-2)+5}{2} & \text{if } i \text{ is odd and } i > 4 \\ \frac{n(i-2)+4}{2} & \text{if } i \text{ is even and } i > 4 \end{cases}$$

where

$$x_i = \begin{cases} v_{\frac{n+2}{2}} & \text{if } i = 1 \\ v_n & \text{if } i = 2 \\ v_{\frac{i-1}{2}} & \text{if } i \text{ is odd and } i > 2 \\ \frac{n+1}{2} + \frac{i}{2} - 1 & \text{if } i \text{ is even and } i > 2 \end{cases}$$

<u>*Proof* :</u>

Note: $diam(P_n) - d(x_i, x_j) + 1 = (n-1) - d(x_i, x_j) + 1 = n - d(x_i, x_j)$
(since $diam(P_n) = n - 1$)
NTS: $\forall i, j \in \{1, 2, ..., n\},\ i \neg j,\ \mid f(x_i) - f(x_j) \mid \geq n - d(x_i, x_j)$

<u>Case 1</u> $i, j \in \{1, 2, 3, 4\}$

<u>Case 1.1</u> $\{i, j\} = \{1, 2\}$
$\mid f(x_1) - f(x_2) \mid = \mid 0 - \frac{n+1}{2} \mid = \mid \frac{n+1}{2} \mid$
$\mid \frac{n+1}{2} \mid \geq \frac{n+1}{2} = n - (n - \frac{n+1}{2}) = n - d(v_{\frac{n+1}{2}}, v_n) = n - d(x_1, x_2)$

$\Rightarrow \mid f(x_1) - f(x_2) \mid \geq n - d(x_1, x_2)$

<u>Case 1.2</u> $\{i, j\} = \{1, 3\}$
$\mid f(x_1) - f(x_3) \mid = \mid 0 - \frac{n+3}{2} \mid = \mid \frac{n+3}{2} \mid$
$\mid \frac{n+3}{2} \mid \geq \frac{n+1}{2} = n - (\frac{n+1}{2} - 1) = n - d(v_{\frac{n+1}{2}}, v_1) = n - d(x_1, x_3)$

$\Rightarrow \mid f(x_1) - f(x_3) \mid \geq n - d(x_1, x_3)$

<u>Case 1.3</u> $\{i, j\} = \{1, 4\}$

$| f(x_1) - f(x_4) |=| 0 - (n+2) |=| n+2 |$
$| n+2 |\geq n-1 = n - (\frac{n+1}{2} + 1 - \frac{n+1}{2}) = n - d(v_{\frac{n+1}{2}}, v_{\frac{n+1}{2}+1}) = n - d(x_1, x_4)$

$\Rightarrow| f(x_1) - f(x_4) |\geq n - d(x_1, x_4)$

<u>Case 1.4</u> $\{i, j\} = \{2, 3\}$
$| f(x_2) - f(x_3) |=| \frac{n+1}{2} - \frac{n+3}{2} |=| -\frac{2}{2} |= 1$
$1 \geq 1 = n - (n-1) = n - d(v_n, v_1) = n - d(x_2, x_3)$

$\Rightarrow| f(x_2) - f(x_3) |\geq n - d(x_2, x_3)$

<u>Case 1.5</u> $\{i, j\} = \{2, 4\}$
$| f(x_2) - f(x_4) |=| \frac{n+1}{2} - (n+2) |=| \frac{-n-3}{2} |=| \frac{n+3}{2} |$
$| \frac{n+3}{2} |\geq \frac{n+3}{2} = n - (n - (\frac{n+1}{2} + 1)) = n - d(v_n, v_{\frac{n+1}{2}+1}) = n - d(x_2, x_4)$

$\Rightarrow| f(x_2) - f(x_4) |\geq n - d(x_2, x_4)$

<u>Case 1.6</u> $\{i, j\} = \{3, 4\}$
$| f(x_3) - f(x_4) |=| \frac{n+3}{2} - (n+2) |=| \frac{-n-1}{2} |=| \frac{n+1}{2} |$
$| \frac{n+1}{2} |\geq \frac{n-1}{2} = n - (\frac{n+1}{2} + 1 - 1) = n - d(v_1, v_{\frac{n+1}{2}+1}) = n - d(x_3, x_4)$

$\Rightarrow| f(x_3) - f(x_4) |\geq n - d(x_3, x_4)$

<u>Case 2</u> $i \in \{1, 2, 3, 4\}$, j is odd and $j > 4$
$\Rightarrow j \geq 5$

<u>Case 2.1</u> $i = 1$
$| f(x_1) - f(x_j) |=| 0 - \frac{n(j-2)+5}{2} |=| \frac{n(j-2)+5}{2} |$
$| \frac{n(j-2)+5}{2} |\geq \frac{n+(j-2)}{2}$
(Since $j \geq 5$ and for any positive integer $z \geq 2$ $nz > n + z$)
$\Rightarrow| \frac{n(j-2)+5}{2} |\geq \frac{n+j-2}{2} = \frac{n+1+j+1}{2} = n - (\frac{n+1}{2} - \frac{j-1}{2}) = n - d(v_{\frac{n+1}{2}}, v_{\frac{j-1}{2}})$
$= n - d(x_1, x_j)$

$\Rightarrow| f(x_1) - f(x_j) |\geq n - d(x_1, x_j)$

<u>Case 2.2</u> $i = 2$
$| f(x_2) - f(x_j) |=| \frac{n+1}{2} - \frac{n(j-2)+5}{2} |=| \frac{nj-3n+4}{2} |$
$| \frac{nj-3n+4}{2} |\geq \frac{j-1}{2}$
(Since $j(n-1) > 3n - 1$ since $j \geq 5$ and $n \geq 3 \Rightarrow\ nj - j \geq 3n - 1$
$\Rightarrow\ nj - 3n \geq j - 1 \Rightarrow\ n(j-3) \geq j - 1$)
$\Rightarrow| \frac{nj-3n+4}{2} |\geq \frac{j-1}{2} = n - (n - \frac{j-1}{2}) = n - d(v_n, v_{\frac{j-1}{2}}) = n - d(x_2, x_j)$

$\Rightarrow| f(x_2) - f(x_j) |\geq n - d(x_2, x_j)$

Case 2.3 $i = 3$
$| f(x_3) - f(x_j) | = | \frac{n+3}{2} - \frac{n(j-2)+5}{2} | = | \frac{nj-3n+2}{2} |$
$| \frac{nj-3n+2}{2} | = | \frac{n(j-3)+2}{2} | \geq \frac{2n-(j-3)}{2} = \frac{2n-j+3}{2}$ (Since $j \geq 5 \ \Rightarrow j-3 \geq 2$)
$\Rightarrow | \frac{nj-3n+2}{2} | \geq \frac{2n-j+3}{2} = n - \frac{j-1}{2} + 1 = n - (\frac{j-1}{2} - 1) = n - d(v_1, v_{\frac{j-1}{2}})$
$= n - d(x_3, x_j)$

$\Rightarrow | f(x_3) - f(x_j) | \geq n - d(x_3, x_j)$

Case 2.4 $i = 4$
$| f(x_4) - f(x_j) | = | n + 2 - \frac{n(j-2)+5}{2} | = | \frac{nj-4n+1}{2} |$
$| \frac{nj-4n+1}{2} | = | \frac{n(j-4)+1}{2} | \geq \frac{n+(j-4)}{2} = \frac{n+j-4}{2}$
(Since $j \geq 5$ and for any positive integer $z \geq 2$ $nz > n + z$)
$\Rightarrow | \frac{nj-4n+1}{2} | \geq \frac{n+j-4}{2} = \frac{n-1}{2} - 1 + \frac{j-1}{2} = n - (\frac{n+1}{2} + 1 - \frac{j-1}{2})$
$= n - d(v_{\frac{n+1}{2}+1}, v_{\frac{j-1}{2}}) = n - d(x_4, x_j)$

$\Rightarrow | f(x_4) - f(x_j) | \geq n - d(x_4, x_j)$

Case 3 $i \in \{1, 2, 3, 4\}$, j is even and $j > 4$
$\Rightarrow j \geq 6$

Case 3.1 $i = 1$
$| f(x_1) - f(x_j) | = | 0 - \frac{n(j-2)+4}{2} | = | \frac{n(j-2)+4}{2} |$
$| \frac{n(j-2)+4}{2} | \geq \frac{2n-j+2}{2}$ (Since $j \geq 6$)
$\Rightarrow | \frac{n(j-2)+4}{2} | \geq \frac{2n-j+2}{2} = n - \frac{j}{2} + 1 = n - ((\frac{n+1}{2} + \frac{j-2}{2}) - \frac{n+1}{2})$
$= n - d(v_{\frac{n+1}{2}}, v_{\frac{n+1}{2}+\frac{j-2}{2}}) = n - d(x_1, x_j)$

$\Rightarrow | f(x_1) - f(x_j) | \geq n - d(x_1, x_j)$

Case 3.2 $i = 2$
$| f(x_2) - f(x_j) | = | \frac{n+1}{2} - \frac{n(j-2)+4}{2} | = | \frac{nj-3n+3}{2} |$
$| \frac{nj-3n+3}{2} | = | \frac{n(j-1)+3}{2} | \geq \frac{n+(j-1)}{2} = \frac{n-j+1}{2}$ (Since $j \geq 6 \Rightarrow j-3 > 0$)
$\Rightarrow | \frac{nj-3n+5}{2} | \geq \frac{n+j-1}{2} = \frac{n+1}{2} + \frac{j-2}{2} = n - (n - (\frac{n+1}{2} + \frac{j-2}{2})) = n - d(v_n, v_{\frac{n+1}{2}+\frac{j-2}{2}})$
$= n - d(x_2, x_j)$

$\Rightarrow | f(x_2) - f(x_j) | \geq n - d(x_2, x_j)$

Case 3.3 $i = 3$
$| f(x_3) - f(x_j) | = | \frac{n+3}{2} - \frac{n(j-2)+4}{2} | = | \frac{nj-3n+1}{2} |$
$| \frac{nj-3n+1}{2} | = | \frac{n(j-3)+1}{2} | \geq \frac{n-(j-3)}{3} \frac{n-j+3}{2}$ (Since $j \geq 6 \Rightarrow j-3 > 0$)
$\Rightarrow | \frac{nj-3n+3}{2} | \geq \frac{n-j+3}{2} = n - \frac{n+1}{2} - \frac{j}{2} + 1 + 1 = n - (\frac{n+1}{2} + \frac{j-2}{2} - 1)$
$= n - d(v_1, v_{\frac{n+1}{2}+\frac{j-2}{2}}) = n - d(x_3, x_j)$

$\Rightarrow \mid f(x_3) - f(x_j) \mid \geq n - d(x_3, x_j)$

<u>Case 3.4</u> $i = 4$

$\mid f(x_4) - f(x_j) \mid = \mid n + 2 - \frac{n(j-2)+4}{2} \mid = \mid \frac{nj-4n}{2} \mid$

$\mid \frac{nj-4n}{2} \mid = \mid \frac{n(j-4)}{2} \mid \geq \frac{2n-(j+4)}{2} = \frac{2n-j-4}{2}$ (Since $j \geq 6 \Rightarrow j - 4 \geq 2$)

$\Rightarrow \mid \frac{nj-4n+2}{2} \mid \geq \frac{2n-j-4}{2} = n - \frac{j}{2} - 2 = n - (\frac{n+1}{2} + \frac{j-2}{2} - \frac{n+1}{2} - 1)$

$= n - d(v_{\frac{n+1}{2}+1}, v_{\frac{n+1}{2}+\frac{j-2}{2}}) = n - d(x_4, x_j)$

$\Rightarrow \mid f(x_4) - f(x_j) \mid \geq n - d(x_4, x_j)$

<u>Case 4</u> $i, j > 4$

<u>Case 4.1</u> i and j are both odd

Without loss of generality, let $i > j$

$\mid f(x_i) - f(x_j) \mid = \mid \frac{n(i-2)+5}{2} - \frac{n(j-2)+5}{2} \mid = \mid \frac{n(i-j)}{2} \mid$

$\mid \frac{n(i-j)}{2} \mid \geq \frac{2n+(j-i)}{2}$ (Since $i > j,\ i - j \geq 2$ and $j - i < 0$)

$\Rightarrow \mid \frac{n(i-j)}{2} \mid \geq \frac{2n+(j-i)}{2} = n - \frac{i-1}{2} + \frac{j-1}{2} = n - (\frac{i-1}{2} - \frac{j-1}{2}) = n - d(v_{\frac{i-1}{2}}, v_{\frac{j-1}{2}})$

$= n - d(x_i, x_j)$

$\Rightarrow \mid f(x_i) - f(x_j) \mid \geq n - d(x_i, x_j)$

<u>Case 4.2</u> i and j are both even

Without loss of generality, let $i > j$

$\mid f(x_i) - f(x_j) \mid = \mid \frac{n(i-2)+4}{2} - \frac{n(j-2)+4}{2} \mid = \mid \frac{n(i-j)}{2} \mid$

$\mid \frac{(i-j)}{2} \mid \geq \frac{2n+(j-i)}{2}$ (Since $i > j,\ i - j \geq 2$ and $j - i < 0$)

$\Rightarrow \mid \frac{n(i-j)}{2} \mid \geq \frac{2n+(j-i)}{2} = n - \frac{i-2}{2} + \frac{j-2}{2} = n - (\frac{n+1}{2} + \frac{i-2}{2} - \frac{n+1}{2} - \frac{j-2}{2})$

$= n - d(v_{\frac{n+1}{2}+\frac{i-2}{2}}, v_{\frac{n+1}{2}+\frac{j-2}{2}}) = n - d(x_i, x_j)$

$\Rightarrow \mid f(x_i) - f(x_j) \mid \geq n - d(x_i, x_j)$

<u>Case 4.3</u> i and j are of different parity

Without loss of generality, let i be odd and j be even

$\mid f(x_i) - f(x_j) \mid = \mid \frac{n(i-2)+5}{2} - \frac{n(j-2)+4}{2} \mid = \mid \frac{n(i-j)+1}{2} \mid$

$\mid \frac{n(i-j)+1}{2} \mid \geq \frac{n+(i-j)}{2}$

(Since if $i > j \Rightarrow \mid \frac{n(i-j)+1}{2} \mid = \frac{n(i-j)+1}{2} > \frac{n+(j-i)}{2}$ since $i - j > 0$

and if $j > i \Rightarrow \mid \frac{n(i-j)+1}{2} \mid = \mid \frac{-n(j-i)+1}{2} \mid = \frac{(j-i)-1}{2} \geq \frac{n+(i-j)}{2}$ since $i - j < 0$)

$\Rightarrow\mid \frac{n(j-i)+1}{2} \mid\geq \frac{n+(i-j)}{2} = \frac{2n-n-1-j+2+i-1}{2} = n - (\frac{n+1}{2} + \frac{j-2}{2} - \frac{i-1}{2}) = n - d(v_{\frac{i-1}{2}}, v_{\frac{n+1}{2}+\frac{j-2}{2}}) = n - d(x_i, x_j)$

$\Rightarrow\mid f(x_i) - f(x_j) \mid\geq n - d(x_i, x_j)$

Adding a Vertex

Now we turn our attention to another problem. What happens to the radio-labeling of a path graph when we add one vertex and one edge to the graph?

Claim: Connecting a vertex x_m to any vertex in the graph P_n will result in a new graph, call it Λ, with a greater span than P_n

(Proof)

Case 1, x_m is connected to either end of P_n(that is, either of the two vertices in P_n with $deg = 1$) and we allow relabeling of vertices in Λ

For simplicity lets consider only the cases with rn(P_n), the reason being that if we can prove our claim for P_n with the smallest span then it follows that our claim will be true for any other radio label of P_n with a greater span than rn(P_n). When connecting a vertex to either end of a P_n graph with rn(P_n) (here we direct our attention to a special case were P_n has its minimum span), and relabeling the vertices to achieve min span of the new graph, note that it turns into a P_{n+1} graph (still a path), the difference is that the new path graph will have a different parity and a higher order. Using the patterns for even paths and odd paths (week one and week two report) we can relabel the new graph to give us rn(P_{n+1}). It is clear that for any integer n rn(P_n)<(P_{n+1}). But for the sake of formality, we show this below (one must reference the equations presented for P_n graphs when n is either even or odd in order to make sense of the following):

Case 1.1, n is even. $n \geq 4$

$f(x_n) = \frac{n}{2}(n) - n + 1 < f(x_{n+1}) = \frac{(n+1)((n+1)-2)+5}{2} = \frac{n^2+4}{2} = \frac{n^2}{2} + 2$

Case 1.2, n is odd. $n \geq 4$

$f(x_n) = \frac{(n-1)^2}{2} + 2 < f(x_{x+1}) = \frac{(n+1)^2}{2}$-$n + 1 = n^2 + n + 1$

Case 1.3, $n = 3$.

$f(x_3) = \frac{3+3}{2} = 3 < f(x_4) = \frac{4}{2}(4)$-4+1=5

Case 1.4, $n = 2$.

$f(x_2) = \frac{2}{2}(2) - 2 + 1 = 1 < f(x_3) = 3$

Case 1.5, $n = 1$.

$f(x_1) = 0 < f(x_2) = 1$

Case 2, x_m is connected to either end of P_n(producing Λ) and relabeling to all vertices in P_n is restricted.

Now consider the case were we connect a vertex to either end of a P_n graph and we forbid relabeling of the set of vertices of the new graph. In other words, we are trying to show that $rn(P_{n+1})>rn(P_n)$ with the restriction that all the labeling's of the vertices in P_n remain static when we connect x_m to one of the ends of P_n(as mentioned earlier, connecting x_m to an end of P_n creates a P_{n+1} graph). Since case 1 shows that the rn(P_{n+1})>rn(P_n), with rn(P_{n+1}) being the minimum span possible for a P_{n+1} graph, it suffices to note that f(x_m)≥rn(P_{n+1})>rn(P_n)(f(V):The set of integers) since the restrictions placed on P_{n+1} in this case can only make f(x_m) greater if not equal to P_{n+1}

Case 3, x_m is connected to any vertex in the graph P_n with exception of its two end vertices(the two vertices whose distance from either one to the other gives the diameter), and relabeling of the ne graph Λ is aloud.

Now lets consider the case were we connect x_m to the graph P_n anywhere in the mid section of the graph and use relabeling of the new graph, call this graph Λ to attain its minimum span. rn(Λ) must be greater than rn(P_n), if the latter is true then when we restrict relabeling of Λ we will be restricted to a new minimum span, namely $minf(x_m)$ which is greater than or equal to the span of rn(Λ) in the case were the latter is obtained by allowing relabeling(Same argument as case two). We need to refer to a theorem from "The Radio Numbers Of All Graphs Of Order n And Diameter $n-2$ by K.Benson, M.Porter, and M.Tomava, namely that the min span of a graph that consist of connecting a vertex x_m to a vertex of P_n, (recall that we named such a graph Λ), is given by the following equation which is presented posthumous to a necessary definition.(Note that the following equation is derived from the case were 1 and not zero is the smallest numerical value that can be assigned to a vertex)

Defenition 1. Let (n, s) ϵZ where $n \geq 4$ and $n-2 \geq$ s $\geq$ 2. The spire graph $S_{n,s}$ is the graph with the vertices (v_1,...,v_n), and edges (v_i, v_{i+1}—$i = 1, 2..., n-2$) together with the edge (v_s,v_n). The vertex v_n is called the spire. Without loss of generality we always assume that $\lfloor \frac{n}{2} \rfloor \geq s$

Now follows the theorem.

THEOREM 1.

(Radio Number of $S_{n,s}$). Let $S_{n,s}$ be a spire graph, where $\lfloor \frac{n}{2} \rfloor \geq s \geq 2$. Then,

$$rn(S_{n,s}) = \begin{cases} 2k^2 - 4k + 2s + 3, & \text{if } n = 2k,\ 2 \leq s \leq k-2 \\ 2k^2 - 2k, & \text{if } n = 2k,\ s = k-1 \\ 2k^2 - 2k + 1, & \text{if } n = 2k,\ s = k \\ 2k^2 - 2k + 2s, & \text{if } n = 2k+1 \end{cases}$$

The above equation gives the minimum span of any graph that consist of a vertex x_m connected to a vertex in the mid section of P_n. Therefore all we need to do is show that rn(P_n)<rn($S_{n+1,s}$).

Case 3.1, n=2k+1

Case 3.1.1 $n > 4$ $(k > 1)$, and $k-2 \geq s \geq 2$

$rn(P_n)$=$\frac{2k^2}{2}$+2=$2k^2$+2, let $t = k$+1 and $p = 2t$, thus we have rn($S_{p,s}$)=$2t^2 - 4t + 2s + 3$, note that the minimum value for s is 2, if the inequality we are trying to prove holds for the case were s is at its minimum then it will hold for all other s. Thus, rn($S_{p,2}$)=$2k^2+3$. Note that $2k^2+3>2k^2+2$, or in other words $rn(P_n) < rn(S_{p,s})$=$rn(S_{n+1,s})$.

Case 3.1.2 n>4($k>1$), and $s=k-1$

$rn(P_n)$=$\frac{2k^2}{2}$+2=$2k^2+2$, let $t=k+1$ and $p=2t$, thus we have rn($S_{p,s}$)= $2k^2+2k$. It is easy to see that $2k^2+2$<$2k^2=2k$ since $k>1$, therefore $rn(P_n)$<$rn(S_{p,s})$=$rn(S_{n+1,s})$.

Case 3.1.3, $n>4(k>1)$, and $s=k$

$rn(P_n)$=$\frac{2k^2}{2}$+2=$2k^2$+2, let $t=k$+1 and $p=2t$, thus we have rn($S_{p,s}$)=2(k^2+ $2k+1)-2(k+1)+1$=$2k^2+2k+1$. Once again it is easy to see that $2k^2+2$<$2k^2+2k+1$ since $k>1$, therefore $rn(P_n)$<$rn(S_{p,s})$=$rn(S_{n+1,s})$

Case 3.1.4, $n=3$, and $k-2\geq s\geq 2$

$rn(P_n)=3$, $rn(S_{4,s})$=$2s+3$. since the minimum value of s is 2 it suffices to prove that the inequality holds for such case. Indeed $3<5$, therefore $rn(P_n)=$ 3<5=$rn(S_{4,s})$=$rn(S_{n+1,s})$.

Case 3.1.5, $n=3$, and $s=k-1$

$rn(P_n)=3$, $rn(S_{4,s})$=4. Obviously $rn(P_n)=3$<4=$rn(S_{4,s})$=$rn(S_{n+1,s})$.

Case 3.1.6, $n=3$, and $s=k$

$rn(P_n)=3$, $rn(S_{4,s})$=5. Obviously $rn(P_n)=3$<5=$rn(S_{4,s})$=$rn(S_{n+1,s})$.

Case 3.1.7, $n=1$, and $k-2\geq s\geq 2$

$rn(P_n)=0$,$rn(S_{2,s})$=$2s+1$. since the minimum value of s is 2 it suffices to prove that the inequality holds for such case. Indeed $0<3$, therefore $rn(P_n)=$ 0<3=$rn(S_{2,s})$=$rn(S_{n+1,s})$.

Case 3.1.8, $n=1$, and $s=k-1$

$rn(P_n)=0$,$rn(S_{2,s})$=0. Here both radio numbers are equal but remember that the theorem we presented earlier uses 1 instead of zero as the smallest numerical value that can be assigned to a vertex, so in reality $rn(S_{2,s})$=1. Thus $rn(S_{2,s})$=$rn(S_{n+1,s})$>$rn(P_n)$. Concourse the latter revelation implies that there should be a 1 added to all the $rn(S_{p,s})$'s in the earlier cases, but we obviously proved the inequality that we are interested in without it and therefore the bringing this curiosity to our attention now only strengthens our hypothesis.

Case 3.1.9, $n=1$, and $s=k$

$rn(P_n)=0$,$rn(S_{2,s})$=1. Of course 1>0, thus $rn(P_n)=0$<1=$rn(S_{2,s})$=$rn(S_{n+1,s})$.

Case 3.2, n=2k (even)

$rn(P_n)$=$\frac{n^2}{2}$-n+1=$2k^2-2k+1$, $rn(S_{n+1,s})$=$rn(S_{2k+1,s})$, now let $2k+1=h$. $rn(S_{h,s})$=$2k^2-2k+2s$. By theorem 1, $\lfloor\frac{n}{2}\rfloor\geq s\geq$2. Since the minimum value for s is 2 it suffices to show that the inequality of interest holds for such case. Thus $rn(S_{h,s})$=$2k^2-2k+2s$=$2k^2-2k+4$>$2k^2-2k+1$=$rn(P_n)$, or in other words $rn(S_{n+1,s})$>$rn(P_n)$.

Thus, rnP_n<$rnS_{n+1,s}$ and therefore any other labeling to graphs of the form Λ will be greater than or equal to rn$S_{n,s}$.

We now conclude that adding a new vertex to a graph of the form P_n by connecting the new vertex to an arbitrary vertex in P_n will result in a graph with a greater span.

Fifth Power Paths

Now we want to look at the radio-number for fifth power path graphs.
We denote a path with n vertices by P_n, where $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} : i = 1, 2, \ldots, n-1\}$. The r^{th} power of a path graph P_n, denoted by P_n^r, is the path graph constructed from P_n by adding edges between vertices of distance r or less apart in P_n. Hence, $V(P_n^5) = V(P_n)$ and $E(P_n^5) = E(P_n) \cup \{v_i v_{i+2} : i = 1, 2, \ldots, n-2\} \cup \{v_i v_{i+3} : i = 1, 2, \ldots, n-3\} \cup \{v_i v_{i+4} : i = 1, 2, \ldots, n-4\} \cup \{v_i v_{i+5} : i = 1, 2, \ldots, n-5\}$. The diameter of P_n^5 is $\lceil \frac{n-1}{5} \rceil$.

Finding the Lower Bound for $rn(P_n^5)$

Proposition 1 For any $u, v \in V(P_n^5)$, we have:

$$d(u,v) \ = \ \left\lceil \frac{d_{P_n}(u,v)}{5} \right\rceil.$$

The *center* of the path graph P_n^5 is defined as the "*middle*" vertex of P_n^5. An odd path P_{2m+1} has only one center v_{m+1}, while an even path P_{2m} has two centers v_m and v_{m+1}. For each vertex $u \in V(P_n)$, the *level* of u, denoted by $L(u)$, is the smallest distance in P_n from u to a center of P_n. For instance, if $n = 2m+1$, then $L(v_1) = m$ and $L(v_{m+1}) = 0$. Denote the levels of a sequence of vertices A by $L(A)$.
If $n = 2m+1$, then

$$L(v_1, v_2, \ldots, v_{2m+1}) = (m, m-1, \ldots, 3, 2, 1, 0, 1, 2, 3, \ldots, m-1, m).$$

If $n = 2m$, then

$$L(v_1, v_2, \ldots, v_{2m}) = (m-1, m-2, \ldots, 3, 2, 1, 0, 0, 1, 2, 3, \ldots, m-2, m-1).$$

Set the *left-vertices* and *right-vertices* as follows:
If $n = 2m+1$, then the left-vertices and right-vertices, respectively are

$$\{v_1, v_2, \ldots, v_m, v_{m+1}\} \text{ and } \{v_{m+1}, v_{m+2}, \ldots, v_{2m}, v_{2m+1}\}.$$

The center v_{m+1} is both a left-vertex and a right-vertex on an odd path.
If $n = 2m$, then the left-vertices and right-vertices, respectively are

$$\{v_1, v_2, \ldots, v_m\} \text{ and } \{v_{m+1}, v_{m+2}, \ldots, v_{2m}\}.$$

If two vertices are both right (or left)-vertices, then we say that they are on the same side; otherwise, they are on the opposite sides. Observe,

Lemma 1 *If n is odd, then for any $u, v \ \in VP_n^5$, we have :*

$$d(u,v) = \begin{cases} \left\lceil \frac{L(u)+L(v)}{5} \right\rceil & \text{if } u \text{ and } v \text{ are on opposite sides} \\ \\ \left\lceil \frac{|L(u)-L(v)|}{5} \right\rceil & \text{if } u \text{ and } v \text{ are on the same sides} \end{cases}$$

$If\ n\ is\ even,\ then\ for\ any\ u,v\ \in VP_n^5, we\ have:$

$$d(u,v) = \begin{cases} \left\lceil \frac{L(u)+L(v)+1}{5} \right\rceil & \text{if } u \text{ and } v \text{ are on opposite sides} \\ \\ \left\lceil \frac{|L(u)-L(v)|}{5} \right\rceil & \text{if } u \text{ and } v \text{ are on the same sides} \end{cases}$$

Lemma 2 *Let P_n^5 be a $fifth$ power path on n vertices where $n \geq 7$ and let $k = \lceil \frac{n-1}{5} \rceil$ i.e. $k = diam(P_n^5)$.*

$$If\ n\ is\ odd\ then\ rn(P_n^5) \geq \begin{cases} \frac{5}{2}k^2+1, & \text{if } n \equiv 1(mod10) \\ \frac{5}{2}k^2+\frac{1}{2}, & \text{if } n \equiv 3(mod10) \\ \frac{5}{2}k^2+\frac{3}{2}, & \text{if } n \equiv 5(mod10) \\ \frac{5}{2}k^2, & \text{if } n \equiv 7(mod10) \\ \frac{5}{2}k^2+1, & \text{if } n \equiv 9(mod10) \end{cases}$$

Proof Let f be a radio-labeling for P_n^5. Re-arrange $V(P_n^5) = \{x_1, x_2, \ldots, x_n\}$ with $0 = f(x_1) < f(x_2) < f(x_3) < \cdots < f(x_n)$. Note that $f(x_n)$ is the span of f.

By definition, $f(x_{i+1}) - f(x_i) \geq k+1-d(x_i, x_{i+1})$ for $1 \leq i \leq n-1$. Summing up these $n-1$ inequalities, we have the following:

$$f(x_n) \geq (n-1)(k+1) - \sum_{i=1}^{n-1} d(x_i, x_{i+1}). \tag{2}$$

To consider the minimal span of all radio labelings of P_n^5 when n is odd, it suffices to maximize the sum $\sum_{i=1}^{n-1} d(x_i, x_{i+1})$ of P_n^5 to minimize the difference from the inequality (1), therefore by lemma 1:

$$\sum_{i=1}^{n-1} d(x_i, x_{x+1}) \ \leq \ \sum_{i=1}^{n-1} \left\lceil \frac{L(x_i)+L(x_{i+1})}{5} \right\rceil.$$

From the inequality above we get:

1) For each i, the equality for $d(x_i, x_{i+1}) \leq \lceil \frac{L(x_i)+L(x_{i+1})}{5} \rceil$ holds only when x_i and x_{i+1} are on the opposite sides, unless one of them is a center; and

2) In the summation $\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \sum_{i=1}^{n-1} \lceil \frac{L(x_i)+L(x_{i+1})}{5} \rceil$, each vertex of P_n^5 occurs exactly twice, except x_i and x_{i+1} for which each only occurs once. Now, consider the following. By direct calculation we have:

$$\left\lceil \frac{L(u)+L(v)}{5} \right\rceil = \begin{cases} \frac{L(u)+L(v)+4}{5} - \frac{4}{5}, & \text{if } L(u)+L(v) \equiv 0 \pmod 5 \\ \frac{L(u)+L(v)+4}{5}, & \text{if } L(u)+L(v) \equiv 1 \pmod 5 \\ \frac{L(u)+L(v)+4}{5} - \frac{1}{5}, & \text{if } L(u)+L(v) \equiv 2 \pmod 5 \\ \frac{L(u)+L(v)+4}{5} - \frac{2}{5}, & \text{if } L(u)+L(v) \equiv 3 \pmod 5 \\ \frac{L(u)+L(v)+4}{5} - \frac{3}{5}, & \text{if } L(u)+L(v) \equiv 4 \pmod 5 \end{cases}$$

Therefore,

$$\left\lceil \frac{L(x_i)+L(x_{i+1})}{5} \right\rceil \leq \frac{L(x_i)+L(x_{i+1})+4}{5}.$$

and the equality holds if $L(u)+L(v) \equiv 1 (mod 5)$. By observation, there exists at most $n-5$ of the i's such that $d(x_i, x_{i+1}) = (L(x_i)+L(x_{i+1})+4)/5$. Furthermore, this concludes the following:

$$\begin{aligned}
\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq & \left[\sum_{i=1}^{n-1} \frac{L(x_i)+L(x_{i+1})+4}{5}\right] - \frac{1}{5} - \frac{1}{5} - \frac{1}{5} - \frac{1}{5} \\
= & \frac{1}{5}\left[\left(2\sum_{i=1}^{n} L(x_i)\right) - L(x_1) - L(x_n)\right] + \frac{4}{5}(n-1) - \frac{4}{5} \\
\leq & \frac{2}{5}\left[2\left(1+2+\cdots+\left(\frac{n-1}{2}\right)\right)\right] - \frac{1}{5} + \frac{4}{5}(n-1) - \frac{4}{5} \quad \left(\text{note } L(x_i)+L(x_n) \geq 1\right) \\
= & \frac{2}{5}\left[\left(\frac{n-1}{2}\right)\left(1+\frac{n-1}{2}\right)\right] + \frac{4}{5}n - \frac{9}{5}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{5}\left(n-1+\frac{n^2}{2} - n + \frac{1}{2}\right) + \frac{4}{5}n - \frac{9}{5} \\
&= \frac{\frac{n^2}{2} - \frac{1}{2}}{5} + \frac{4}{5}n - \frac{9}{5} \\
&= \frac{n^2}{10} + \frac{4}{5}n - \frac{19}{10}
\end{aligned}$$

Hence, when n is odd, $n \geq 7$,

$$\mathrm{rn}(P_n^5) \geq (n-1)(k+1) - \left(\frac{n^2}{10} + \frac{4}{5}n - \frac{19}{10}\right)$$

Now we must consider 5 cases according to $n \pmod{10}$ when n is odd. By direct calculation and considering that $rn(P_n^5)$ is an integer, we have:

<u>*Case* 1 :</u>

$$\begin{aligned}
&((5k+1)-1)(k+1) - \left(\frac{(5k+1)^2}{10} + \frac{4}{5}(5k+1) - \frac{19}{10} \right) \\
&= \; 5k^2 + 5k - \left(\frac{(25k^2+10k+1)}{10} + 4k + \frac{4}{5} - \frac{19}{10} \right) \\
&= \; \frac{5}{2}k^2 - \frac{1}{10} - \frac{4}{5} + \frac{19}{10} \\
&= \; \left\lceil \frac{5}{2}k^2 + 1 \right\rceil = \frac{5}{2}k^2 + 1, \text{ when } k \text{ is even.}
\end{aligned}$$

<u>*Case* 2 :</u>

$$\begin{aligned}
&((5k-2)-1)(k+1) - \left(\frac{(5k-2)^2}{10} + \frac{4}{5}(5k-2) - \frac{19}{10} \right) \\
&= \; 5k^2 + 2k - 3 - \left(\frac{(25k^2-20k+4)}{10} + 4k - \frac{8}{5} - \frac{19}{10} \right) \\
&= \; \frac{5}{2}k^2 - \frac{4}{10} - \frac{30}{10} + \frac{16}{10} + \frac{19}{10} \\
&= \; \left\lceil \frac{5}{2}k^2 + \frac{1}{10} \right\rceil = \frac{5}{2}k^2 + \frac{1}{2}, \text{ when } k \text{ is odd.}
\end{aligned}$$

Case 3 :

$$
\begin{aligned}
&((5k)-1)(k+1)-\left(\frac{(5k)^2}{10}+\frac{4}{5}(5k)-\frac{19}{10}\right)\\
=\;& 5k^2+4k-1-\left(\frac{(25k^2)}{10}+4k-\frac{19}{10}\right)\\
=\;& \frac{5}{2}k^2-1+\frac{19}{10}\\
=\;& \left\lceil\frac{5}{2}k^2+\frac{9}{10}\right\rceil=\frac{5}{2}k^2+\frac{3}{2},\ \text{when } k \text{ is odd.}
\end{aligned}
$$

Case 4 :

$$
\begin{aligned}
&((5k-3)-1)(k+1)-\left(\frac{(5k-3)^2}{10}+\frac{4}{5}(5k-3)-\frac{19}{10}\right)\\
=\;& 5k^2+k-4-\left(\frac{(25k^2-30k+9)}{10}+4k-\frac{12}{5}-\frac{19}{10}\right)\\
=\;& \frac{5}{2}k^2-4-\frac{4}{10}+\frac{12}{5}+\frac{19}{10}\\
=\;& \left\lceil\frac{5}{2}k^2-\frac{1}{10}\right\rceil=\frac{5}{2}k^2,\ \text{when } k \text{ is even.}
\end{aligned}
$$

Case 5 :

$$
\begin{aligned}
&((5k-1)-1)(k+1)-\left(\frac{(5k-1)^2}{10}+\frac{4}{5}(5k-1)-\frac{19}{10}\right)\\
=\;& 5k^2+3k-2-\left(\frac{(25k^2-10k+1)}{10}+4k-\frac{4}{5}-\frac{19}{10}\right)\\
=\;& \frac{5}{2}k^2-\frac{20}{10}-\frac{1}{10}+\frac{4}{5}+\frac{19}{10}\\
=\;& \left\lceil\frac{5}{2}k^2+\frac{3}{5}\right\rceil=\frac{5}{2}k^2+1,\ \text{when } k \text{ is even.}
\end{aligned}
$$

Therefore, we reach the following:

$$
rn(P_n^5)\geq\begin{cases}
\lceil\frac{5}{2}k^2+1\rceil=\frac{5}{2}k^2+1, & \text{if } n\equiv 1 \pmod{10} \text{ (i.e., n=5k+1 is even);}\\
\lceil\frac{5}{2}k^2+\frac{1}{10}\rceil=\frac{5}{2}k^2+\frac{1}{2}, & \text{if } n\equiv 3 \pmod{10} \text{ (i.e., n=5k-2 is odd);}\\
\lceil\frac{5}{2}k^2+\frac{9}{10}\rceil=\frac{5}{2}k^2+\frac{3}{2}, & \text{if } n\equiv 5 \pmod{10} \text{ (i.e., n=5k is odd);}\\
\lceil\frac{5}{2}k^2-\frac{3}{5}\rceil=\frac{5}{2}k^2, & \text{if } n\equiv 7 \pmod{10} \text{ (i.e., n=5k-3 is even);}\\
\lceil\frac{5}{2}k^2+\frac{3}{5}\rceil=\frac{5}{2}k^2+1, & \text{if } n\equiv 9 \pmod{10} \text{ (i.e., n=5k-1 is even).}
\end{cases}
$$

Now we will look at P_n^5 when n is even to establish the lower bound of $rn(P_n^5)$.
Lemma 3 *Let P_n^5 be a $fifth$ power path on n vertices where $n \geq 7$ and let $k = \lceil \frac{n-1}{5} \rceil$ i.e. $k = diam(P_n^5)$.*

$$If\ n\ is\ even\ then\ rn(P_n^5) \geq \begin{cases} \frac{5}{2}k^2 + 1, & \text{if } n \equiv 0(mod10) \\ \frac{5}{2}k^2 - \frac{1}{2}, & \text{if } n \equiv 2(mod10) \\ \frac{5}{2}k^2 + \frac{1}{2}, & \text{if } n \equiv 4(mod10) \\ \frac{5}{2}k^2 + \frac{3}{2}, & \text{if } n \equiv 6(mod10) \\ \frac{5}{2}k^2, & \text{if } n \equiv 8(mod10) \end{cases}$$

Proof
To consider the minimal span of all radio labelings of P_n^5 when n is even, the situation is very similar to that of the fifth power of odd paths. By lemma 1:

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \ \leq \ \sum_{i=1}^{n-1} \left\lceil \frac{L(x_i) + L(x_{i+1}) + 1}{5} \right\rceil.$$

From the inequality above we get:
1) For each i, the equality for $d(x_i, x_{i+1}) \leq \lceil \frac{L(x_i)+L(x_{i+1}+1)}{5} \rceil$ holds only when x_i and x_{i+1} are on the opposite sides, unless one of them is a center; and
2) In the summation $\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \sum_{i=1}^{n-1} \lceil \frac{L(x_i)+L(x_{i+1}+1)}{5} \rceil$, each vertex of P_n^5 occurs exactly twice, except x_i and x_{i+1} for which each only occurs once. Now, consider the following. By direct calculation we have:

$$\left\lceil \frac{L(u) + L(v) + 1}{5} \right\rceil = \begin{cases} \frac{L(u)+L(v)+5}{5}, & \text{if } L(u) + L(v) \equiv 0 \text{ (mod 5)} \\ \frac{L(u)+L(v)+5}{5} - \frac{1}{5}, & \text{if } L(u) + L(v) \equiv 1 \text{ (mod 5)} \\ \frac{L(u)+L(v)+5}{5} - \frac{2}{5}, & \text{if } L(u) + L(v) \equiv 2 \text{ (mod 5)} \\ \frac{L(u)+L(v)+5}{5} - \frac{3}{5}, & \text{if } L(u) + L(v) \equiv 3 \text{ (mod 5)} \\ \frac{L(u)+L(v)+5}{5} - \frac{4}{5}, & \text{if } L(u) + L(v) \equiv 4 \text{ (mod 5)} \end{cases}$$

Therefore,

$$\left\lceil \frac{L(x_i) + L(x_{i+1}) + 1}{5} \right\rceil \leq \frac{L(x_i) + L(x_{i+1}) + 5}{5}.$$

and the equality holds when $L(u) + L(v) \equiv 0(mod5)$. By observation, there exists at most $n - 5$ of the i's such that $d(x_i, x_{i+1}) = (L(x_i) + L(x_{i+1}) + 5)/5$.

Furthermore, this concludes the following:

$$\begin{aligned}
\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq & \left[\sum_{i=1}^{n-1} \frac{L(x_i) + L(x_{i+1}) + 5}{5}\right] - \frac{1}{5} - \frac{1}{5} - \frac{1}{5} - \frac{1}{5} \\
= & \frac{1}{5}\left[\left(2\sum_{i=1}^{n-1} L(x_i)\right) - L(x_1) - L(x_n)\right] + (n-1) - \frac{4}{5} \\
\leq & \frac{2}{5}\left[2\left(0 + 1 + 2 + \cdots + \left(\frac{n}{2} - 1\right)\right)\right] + n - \frac{9}{5} \ \left(\text{note } L(x_i) = L(x_n) = 0\right) \\
= & \frac{2}{5}\left[\left(1 + \left(\frac{n}{2} - 1\right)\right)\left(\frac{n}{2} - 1\right)\right] + n - \frac{9}{5} \\
= & \frac{n^2}{10} + \frac{4}{5}n - \frac{9}{5}
\end{aligned}$$

Hence when n is even, $n \geq 8$

$$\mathrm{rn}(P_n^5) \ \geq \ (n-1)(k+1) - \left(\tfrac{n^2}{10} + \tfrac{4}{5}n - \tfrac{9}{5}\right)$$

Now we must consider 5 cases according to $n \pmod{10}$ when n is even. By direct calculation and considering that $rn(P_n^5)$ is an integer, we have:

<u>*Case* 1 :</u>

$$
\begin{aligned}
&((5k)-1)(k+1)-\left(\frac{(5k)^2}{10}+\frac{4}{5}(5k)-\frac{9}{5}\right)\\
&= 5k^2+4k-1-\left(\frac{(25k^2)}{10}+4k-\frac{9}{5}\right)\\
&= \frac{5}{2}k^2-1+\frac{9}{5}\\
&= \left\lceil\frac{5}{2}k^2+\frac{4}{5}\right\rceil=\frac{5}{2}k^2+1,\ \text{when } k \text{ is even.}
\end{aligned}
$$

<u>*Case* 2 :</u>

$$
\begin{aligned}
&((5k-3)-1)(k+1)-\left(\frac{(5k-3)^2}{10}+\frac{4}{5}(5k-3)-\frac{9}{5}\right)\\
&= 5k^2+k-4-\left(\frac{(25k^2-30k+9)}{10}+4k-\frac{12}{5}-\frac{9}{5}\right)\\
&= \frac{5}{2}k^2-4-\frac{9}{10}+\frac{12}{5}+\frac{9}{5}\\
&= \left\lceil\frac{5}{2}k^2-\frac{7}{10}\right\rceil=\frac{5}{2}k^2-\frac{1}{2},\ \text{when } k \text{ is odd.}
\end{aligned}
$$

<u>*Case* 3 :</u>

$$
\begin{aligned}
&((5k-1)-1)(k+1)-\left(\frac{(5k-1)^2}{10}+\frac{4}{5}(5k-1)-\frac{19}{10}\right)\\
&= 5k^2+3k-2-\left(\frac{(25k^2-10k+1)}{10}+4k-\frac{4}{5}-\frac{9}{5}\right)\\
&= \frac{5}{2}k^2-2-\frac{1}{10}+\frac{4}{5}+\frac{9}{5}\\
&= \left\lceil\frac{5}{2}k^2+\frac{1}{2}\right\rceil=\frac{5}{2}k^2+\frac{1}{2},\ \text{when } k \text{ is odd.}
\end{aligned}
$$

Case 4 :

$$\begin{aligned}
&((5k+1)-1)(k+1) - \left(\frac{(5k+1)^2}{10} + \frac{4}{5}(5k+1) - \frac{9}{5}\right)\\
&= 5k^2 + 5k - \left(\frac{(25k^2+10k+1)}{10} + 4k + \frac{4}{5} - \frac{9}{5}\right)\\
&= \frac{5}{2}k^2 + \frac{9}{10}\\
&= \left\lceil \frac{5}{2}k^2 + \frac{9}{10} \right\rceil = \frac{5}{2}k^2 + \frac{3}{2}, \text{ when } k \text{ is odd.}
\end{aligned}$$

Case 5 :

$$\begin{aligned}
&((5k-2)-1)(k+1) - \left(\frac{(5k-2)^2}{10} + \frac{4}{5}(5k-2) - \frac{9}{5}\right)\\
&= 5k^2 + 2k - 3 - \left(\frac{(25k^2-20k+4)}{10} + 4k - \frac{8}{5} - \frac{9}{5}\right)\\
&= \frac{5}{2}k^2 - 3\frac{2}{5} - \frac{1}{10} + \frac{8}{5} + \frac{9}{5}\\
&= \left\lceil \frac{5}{2}k^2 \right\rceil = \frac{5}{2}k^2, \text{ when } k \text{ is even.}
\end{aligned}$$

Therefore, we reach the following:

$$rn(P_n^5) \geq \begin{cases} \lceil \frac{5}{2}k^2 + \frac{4}{5} \rceil = \frac{5}{2}k^2 + 1, & \text{if } n \equiv 0 \pmod{10} \text{ (i.e., n=5k is even);} \\ \lceil \frac{5}{2}k^2 - \frac{1}{2} \rceil = \frac{5}{2}k^2 - \frac{1}{2}, & \text{if } n \equiv 2 \pmod{10} \text{ (i.e., n=5k-4 is odd);} \\ \lceil \frac{5}{2}k^2 + \frac{1}{2} \rceil = \frac{5}{2}k^2 + \frac{1}{2}, & \text{if } n \equiv 4 \pmod{10} \text{ (i.e., n=5k-1 is odd);} \\ \lceil \frac{5}{2}k^2 + \frac{3}{2} \rceil = \frac{5}{2}k^2 + \frac{3}{2}, & \text{if } n \equiv 6 \pmod{10} \text{ (i.e., n=5k+1 is odd);} \\ \lceil \frac{5}{2}k^2 \rceil = \frac{5}{2}k^2, & \text{if } n \equiv 8 \pmod{10} \text{ (i.e., n=5k-2 is even).} \end{cases}$$

Therefore, by combining lemma 2 and lemma 3, we obtained a "general" lower bound for $rn(P_n^5)$:

Lemma 4: *Let P_n^5 be a $fifth$ power path on n vertices where $n \geq 7$ and let $k = \lceil \frac{n-1}{5} \rceil$ i.e. $k = diam(P_n^5)$.*

$$rn(P_n^5) \geq \begin{cases} \frac{5}{2}k^2 + 1, & \text{if } n \equiv 0,1,9 \pmod{10} \\ \frac{5}{2}k^2 - \frac{1}{2}, & \text{if } n \equiv 2 \pmod{10} \\ \frac{5}{2}k^2 + \frac{1}{2}, & \text{if } n \equiv 3,4 \pmod{10} \\ \frac{5}{2}k^2 + \frac{3}{2}, & \text{if } n \equiv 5,6 \pmod{10} \\ \frac{5}{2}k^2, & \text{if } n \equiv 7,8 \pmod{10} \end{cases}$$

However, some of the cases do not follow the "general lower-bound" and we must prove that the lower bound for those cases is actually sharper.

Proof that the Radio Number of P^5_{10q+8} Must be Raised

First, consider the level of a vertex in respect to the graph's center. The levels of the path graph P^5_{10q+8} show that there are extra vertices with a level value of 1 that must be placed accordingly to achieve a minimum span. Since a pattern without jumping is conclusively what we're seeking, the radio number is restricted to a pattern based on our previous findings of the lower bound, which gives the sum of levels of two vertices to be congruent to 0, 1, or 2 as our best cases.
We know that there are $n-1$ connections (edges) on a path graph. Therefore the following must be true:

$$n - 1 = 10q + 8n - 9 = 10q$$

Therefore there exists no more than $n - 9 = 10q$ connections for P^5_{10q+8}.
For our lower bound pattern to P^5_n, we used notation based on whether vertices are on the "left" or on the "right" side of a path graph. Similarly, for the levels of this graph, we consider the equivalent form of levels. For m = left, and r = right, we have the following pattern:

$$\begin{array}{c} m_0 <----> r_0 \\ m_1 <----> r_4 \\ m_2 <----> r_3 \\ m_3 <----> r_2 \\ m_4 <----> r_1 \end{array}$$

In P^5_{10q+8}, there exists given amounts for each level for each side respectively. For example, there are at least $q + 1$ many m_0 and r_0 in the graph. By adding these together and subtracting 1, you get the amount of connections between m_0 and r_0 in the given pattern. Accordingly, you do this for each level to reach the following:

$$\begin{array}{l} 2q+1 \text{ connections for } m_0 <----> r_0 \\ 2q+1 \text{ connections for } m_1 <----> r_4 \\ 2q+1 \text{ connections for } m_2 <----> r_3 \\ 2q+1 \text{ connections for } m_3 <----> r_2 \\ 2q-1 \text{ connections for } m_4 <----> r_1 \end{array}$$

With these connections known, then there exists disconnections in the pattern since we must continue to assign an appropriate radio labeling. By inspection, there is at least 4 disconnections in P^5_n since the pattern must go from $r_0 \Rightarrow m_1$, $r_4 \Rightarrow m_2$, $r_3 \Rightarrow m_3$, and $r_2 \Rightarrow m_4$. By using the above conclusions, with enough disconnections, the levels of the graph will force the radio number to bump, allowing a successful pattern to work.

Under these assumptions, we can prove that the lower bound must be bumped by 1. We'll use the same method to prove the other remaining cases in the Fall Quarter of 2012.

Upper Bound and Optimal Radio-labelings for $rn(P_n^5)$

By Lemma 4, to establish $rn(P_n^5)$, it suffices to give a radio-labeling that gives us the desired span. We will use Lemma 5 to show that a given labeling is a radio-labeling.

Lemma 5

Let P_n^5 be a fifth power path graph on n vertices with $k = \lceil \frac{n-1}{5} \rceil$ i.e. $k = diam(P_n^5)$. Let $\{x_1, x_2, x_3, ..., x_n\}$ be a permutation of $V(P_n^5)$ s.t. for any $1 \leq i \leq n-2$:

$$min\ \{d_{P_n}(x_i, x_{i+1}), d_{P_n}(x_{i+1}, x_{i+2})\}$$

$\leq \frac{5}{2}k + 1$ and $max\ \{d_{P_n}(x_i, x_{i+1}), d_{P_n}(x_{i+1}, x_{i+2})\} \equiv 0,\ 2,$ or $4 (\text{mod } 5)$ if k is even and the equality above holds. Let f be a function $f \longrightarrow \{0, 1, 2, ...\}$ with $f(x_1) = 0$ and $f(x_{i+1}) - f(x_i) = k + 1 - d(x_i, x_{i+1})$ for all $1 \leq i \leq n-1$. Then f is a radio labeling for P_n^5.

Proposition 1

For any d_1 $d_2 \in \mathbb{N}$ we have,

$$\lceil \frac{d_1 + d_2}{5} \rceil = \begin{cases} \lceil \frac{d_1}{5} \rceil + \lceil \frac{d_2}{5} \rceil - 1 \text{ if } (d_1, d_2) \equiv (1,1), (1,2), (2,1), (1,3), (3,1), \\ (1,4), (4,1), (2,2), (2,3), \text{ or } (3,2) (mod\ 5) \\ \lceil \frac{d_1}{5} \rceil + \lceil \frac{d_2}{5} \rceil \text{ otherwise} \end{cases}$$

$$\lceil \frac{d_1 - d_2}{5} \rceil = \begin{cases} \lceil \frac{d_1}{5} \rceil - \lceil \frac{d_2}{5} \rceil + 1 \text{ if } (d_1, d_2) \equiv (0,1), (0,2), (0,3), (0,4), (2,1), (3,1), \\ (4,1), (3,2), (4,2), \text{ or } (4,3) (mod\ 5) \\ \lceil \frac{d_1}{5} \rceil - \lceil \frac{d_2}{5} \rceil \text{ otherwise} \end{cases}$$

Proof of Lemma :

Let f be a function satisfying the assumption. It suffices to prove that $f(x_j) - f(x_i) \geq k + 1 - d(x_i, x_j)$ for any $j \geq i + 2$. For $i = 1, 2, ..., n-1$, set

$f_i = f(x_{i+1}) - f(x_i)$

For any $j \geq i+2$ it follows that $f(x_j) - f(x_i) = f_i + f_{i+1} + f_{i+2} + ... + f_{j-1}$

<u>Case 1</u> $j = i + 2$

Assume $d(x_i, x_{i+1}) \geq d(x_{i+1}, x_{i+2})$ (the proof for $d(x_i, x_{i+1}) \leq d(x_{i+1}, x_{i+2})$ is similar.) Then

$d(x_{i+1}, x_{i+2}) \leq \lceil \frac{\frac{5}{2}k+1}{5} \rceil \leq \begin{cases} \frac{k+1}{2} & \text{if } \; k \text{ is odd} \\ \frac{k+2}{2} & \text{if } \; k \text{ is even} \end{cases}$

and therefore, $d(x_{i+1}, x_{i+2}) \leq \frac{k+2}{2}$. It suffices to consider the following subcases

Case 1.1 x_i is between x_{i+1} and x_{i+2}
Then $d(x_{i+1}, x_{i+2}) \geq d(x_i, x_{i+1})$. Since we assume $d(x_i, x_{i+1}) \geq d(x_{i+1}, x_{i+2})$, we have $d(x_i, x_{i+1}) = d(x_{i+1}, x_{i+2}) \leq \frac{k-2}{2}$ and $d_{P_n}(x_i, x_{i+2}) \leq 2$ from which we have $d(x_i, x_{i+2}) = 1$. Hence,

$$\begin{aligned} f(x_{i+2}) - f(x_i) &= f_i + f_{i+1} \\ &= k + 1 - d(x_i, x_{i+1}) + k + 1 - d(x_{i+1}, x_{i+2}) \\ &\geq 2k + 2 - 2(\frac{k+2}{2}) \\ &= k + 1 - d(x_i, x_{i+2}) \end{aligned}$$

Case 1.2 x_{i+1} is between x_i and x_{i+2}
This implies
$d(x_i, x_{i+2}) = \lceil \frac{d_{P_n}(x_i,x_{i+1})+d_{P_n}(x_{i+1},x_{i+2})}{5} \rceil \geq d(x_i, x_{i+1}) + d_{P_n}(x_{i+1}, x_{i+2}) - 1$
by Lemma 1.
Similar to the calculations above, we have $f(x_{i+2}) - f(x_i) \geq k + 1 - d(x_i, x_{i+2})$

Case 1.3 x_{i+2} is between x_i and x_{i+1}
Assume k is odd or $min \ \{d_{P_n}(x_i, x_{i+1}), d_{P_n}(x_{i+1}, x_{i+2})\} \leq \frac{5}{2}k$, then we have $d(x_{i+1}, x_{i+2}) \leq \frac{k+1}{2}$ and

$d(x_i, x_{i+2}) = \lceil \frac{d_{P_n}(x_i,x_{i+1})-d_{P_n}(x_{i+1},x_{i+2})}{5} \rceil \geq d_{P_n}(x_i, x_{i+1}) - d_{P_n}(x_{i+1}, x_{i+2})$
by Lemma 1.
Hence, $f(x_{i+2}) - f(x_i) \geq k + 1 - d(x_i, x_{i+2})$

If k is even and $min \ \{d_{P_n}(x_i, x_{i+1}), d_{P_n}(x_{i+1}, x_{i+2})\} = \frac{5}{2}k + 1$ then by our assumption it must be that $d_{P_n}(x_{i+1}, x_{i+2}) = \frac{5}{2}k + 1 \equiv 1$ or $3(mod 5)$ and $d_{P_n}(x_i, x_{i+1}) \equiv 0, 2,$ or $4(mod 5)$. Thus we have,

$d(x_i, x_{i+2}) = d(x_i, x_{i+1}) - d(x_{i+1}, x_{i+2}) + 1$

which implies

$f(x_{i+2}) - f(x_i) = 2k + 2 - (d(x_i, x_{i+2}) - d(x_i, x_{i+2}) - 1) - d(x_{i+1}, x_{i+2}) \geq 2k + 3 - d(x_i, x_{i+2}) - 2(\frac{k+2}{2}) = k + 1 - d(x_i, x_{i+2})$

Case 2 $j = i + 3$

Case 2.1
The sum of some pair of the distances $d(x_i, x_{i+1}), d(x_{i+1}, x_{i+2})$, and $d(x_{i+2}, x_{i+3})$

is at most $k+2$. Then

$f(x_{i+3}) - f(x_i) = 3k+3 - d(x_i, x_{i+1}) - d(x_{i+1}, x_{i+2}) - d(x_{i+2}, x_{i+3}) \geq 3k+3-(k+2)-k > k+1-d(x_i, x_{i+3})$

<u>Case 2.2</u>

The sum of any pair of distances $d(x_i, x_{i+1})$, $d(x_{i+1}, x_{i+2})$, and $d(x_{i+2}, x_{i+3})$ is greater than $k+2$. If we assume further that $d(x_i, x_{i+1}) \geq d(x_{i+1}, x_{i+2})$ (the proof for $d(x_{i+1}) \geq d(x_{i+1}, x_{i+2})$ is similar), from the calculation in case 1, we have $d(x_{i+1}, x_{i+2}) \leq \frac{k+2}{2}$. By our hypnosis it follows that $d(x_i, x_{i+1})$ and $d(x_{i+2}, x_{i+3})$ must both be greater than $\frac{k+2}{2}$. This result together with $diam(P_n^5) = k$ and our assumption that the sum of any pair of distances $d(x_i, x_{i+1})$, $d(x_{i+1}, x_{i+2})$, and $d(x_{i+2}, x_{i+3})$ is greater than $k+2$, x_i must appear before x_{i+2}, then x_{i+1}, then x_{i+3}, from left to right on the fifth power path (or x_{i+3} must appear before x_{i+1}, then x_{i+2}, then x_i). Therefore,

$d(x_i, x_{i+3}) \geq d(x_i, x_{i+1}) + d(x_{i+2}, x_{i+3}) - d(x_{i+1}, x_{i+2}) - 1$

Therefore, we have

$f(x_{i+3}) - f(x_i) = 3k+3-d(x_i, x_{i+1})-d(x_{i+1}, x_{i+2})-d(x_{i+2}, x_{i+3}) \geq 3k+3-d(x_i, x_{i+3})-2d(x_{i+1}, x_{i+2})-1 \geq 3k+3-d(x_i, x_{i+3})-2(\frac{k+2}{2}) \geq k+1-d(x_i, x_{i+3})$

<u>Case 3</u> $j \geq i+4$

Since $min\ \{d_{P_n}(x_i, x_{i+1}), d_{P_n}(x_{i+1}, x_{i+2})\} \leq \frac{k+2}{2}$ and $f_i \geq k+1-d(x_i, x_{i+1})$ for any i, we have $max\ \{f_i, f_{i+1}\} \geq \frac{k}{2}$ for any $1 \leq i \leq n-2$

Therefore

$f(x_j)-f(x_i) \geq (f_i+f_{i+1})+(f_{j+2}+f_{j+3}) \geq (\frac{k}{2}+1)+(\frac{k}{2}+1) > k+1-d(x_i, x_j)$

Using Lemma 5, it will now be easy to show that the labelings we found are radio-labelings.
To show the existence of a radio-labeling of P_n^5 achieving the desired lower bound, we consider the cases separately according to n(mod 10). For each desired radio-labeling, f given the following, we shall first define a permutation(line-up) of the vertices $V(P_n^5) = \{x_1, x_2, ..., x_n\}$ then define f by $f(x_1) = 0$ and for all $1 \leq i \leq n-1$, $f(x_{i+1}) = f(x_i) + k + 1 - d(x_i, x_{i+1})$.

<u>Case 1</u> $n \equiv 0(mod 10)$ $(rn(P_n^5) \leq \frac{5}{2}k^5 + 1)$

When $n \equiv 0$ (mod 10) $\Rightarrow \exists\ q \in \mathbb{N}$ s.t. $n = 10q \Rightarrow k = diam(P_n^5) = 2q$. So $n = 10, 20, 30, \ldots$. For this case we were able to find a pattern where the span matched the general lower bound found earlier. The easiest way to describe the pattern we found is by looking at the levels of the vertices.

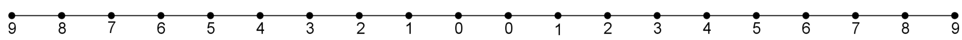

Figure 12: The levels of the vertices of P_{20}

Take P_{20}^5 for example, we start with the left vertex with level 0. Then we move to the right vertex with level 5 followed by the left vertex with level 5. Next we

go to the right vertex with level 1 followed by the left vertex with level 9. Then we go to the right vertex with level 6 followed by the left vertex with level 4, etc.... So the pattern for labeling P_{20}^5 looks as follows:
L0-R5-L5-R1-L9-R6-L4- R2-L8-R7-L3-R3-L7-R8-L2-R4-L6-R9-L1-R0
Where Ll represents a left vertex with level l and Rl represents a right vertex with level l. After close examination, we noticed that the pattern for P_{20}^5 can be re-written as:
L0-R$(5q-5)$-L5-R1-L$(5q-1)$-R6-L$(5q-6)$-R2-L$(5q-2)$-R7-L$(5q-7)$-R3-L$(5q-$
$3)$
-R8-L$(5q-8)$-R4-L$(5q-4)$-R9-L1-R0.
We also found that this pattern can be extended for all P_n^5 when $n = 10q$. So in general the pattern is:

$$
\begin{aligned}
&L0 \xrightarrow{5q-4} R(5q-5) \xrightarrow{5q+1} L5 \xrightarrow{5q-4} R(5q-10) \xrightarrow{5q+1} L10 \xrightarrow{5q-4} R(5q-15) \xrightarrow{5q+1} L15 \xrightarrow{5q-4} \ldots \xrightarrow{5q-4} R10 \xrightarrow{5q+1} L(5q-10) \xrightarrow{5q-4} R5 \xrightarrow{5q+1} L(5q-5) \\
&\xrightarrow{5q-3} R1 \xrightarrow{5q+1} L(5q-1) \xrightarrow{5q+6} R6 \xrightarrow{5q+1} L(5q-6) \xrightarrow{5q+6} R11 \xrightarrow{5q+1} L(5q-11) \xrightarrow{5q+6} \ldots \xrightarrow{5q+6} R(5q-9) \xrightarrow{5q+1} L9 \xrightarrow{5q+6} R(5q-4) \xrightarrow{5q+1} L4 \\
&\xrightarrow{7} R2 \xrightarrow{5q+1} L(5q-2) \xrightarrow{5q+6} R7 \xrightarrow{5q+1} L(5q-7) \xrightarrow{5q+6} R12 \xrightarrow{5q+1} L(5q-12) \xrightarrow{5q+6} \ldots \xrightarrow{5q+6} R(5q-8) \xrightarrow{5q+1} L8 \xrightarrow{5q+6} R(5q-3) \xrightarrow{5q+1} L3 \\
&\xrightarrow{7} R3 \xrightarrow{5q+1} L(5q-3) \xrightarrow{5q+6} R8 \xrightarrow{5q+1} L(5q-8) \xrightarrow{5q+6} R13 \xrightarrow{5q+1} L(5q-13) \xrightarrow{5q+6} \ldots \xrightarrow{5q+6} R(5q-7) \xrightarrow{5q+1} L7 \xrightarrow{5q+6} R(5q-2) \xrightarrow{5q+1} L2 \\
&\xrightarrow{7} R4 \xrightarrow{5q+1} L(5q-4) \xrightarrow{5q+6} R9 \xrightarrow{5q+1} L(5q-9) \xrightarrow{5q+6} R14 \xrightarrow{5q+1} L(5q-14) \xrightarrow{5q+6} \ldots \xrightarrow{5q+6} R(5q-6) \xrightarrow{5q+1} L6 \xrightarrow{5q+6} R(5q-1) \xrightarrow{5q+1} L1 \xrightarrow{2} R0
\end{aligned}
$$

Thus $x_1 = L0$, $x_2 = R(5q-5)$,$x_3 = L5$,$\ldots$,$x_n = x_{10q} = R0$. The values above the arrow show the distances between the two consecutive vertices. By Lemma 5, f is a radio-labeling for P_{10q}^5. Observe from above, there are five possible distances in P_{10q}^5 between consecutive vertices, 1, 2, q, $q+1$, and $q+2$, with the number of occurrences 1, 3, q, $5q-1$, and $4q-4$ respectively. It follows by direct calculation (note that $q = \frac{k}{2}$) that

$$f(x_{10q}) = (10q-1)(k+1) - \sum_{i=1}^{10q-1} d(x_i, x_{i+1}) = \frac{5}{2}k^2 + 1$$

As an example, the following figures show our pattern for the first two cases for $n = 10q$:

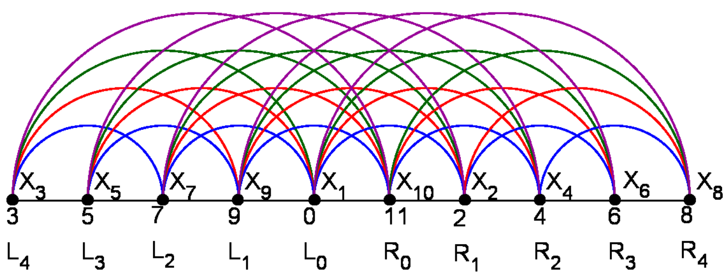

Figure 13: A radio-labeling of P_{10}^5

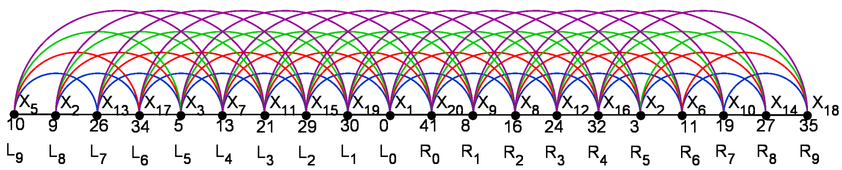

Figure 14: A radio-labeling of P_{20}^5

<u>Case 2</u> $n \equiv 9 (\text{mod } 10)$ $(rn(P_n^5) \leq \frac{5}{2}k^5 + 1)$
When $n \equiv 9$ (mod 10) $\Rightarrow \exists\, q \in \mathbb{N}$ s.t. $n = 10q - 1 \Rightarrow k = diam(P_n^5) = 2q$. So $n = 9, 19, 29, 39, \ldots$. Let $G = P_{10q}^5$ and H be the subgraph of G induced by the vertex set $v_1, v_2, v_3, ..., v_{10q-1}$. Then $H \cong P_{10q-1}^5$, $diam(H) = diam(G) = 2q$, and $d_G(u, v) = d_H(u, v)$ for every $u, v \in V(H)$. Let f be a radio-labeling for G, then $f\mid_H$ is also a radio-labeling for H. By Case 1, $rn(P_{10q-1}^5) \leq rn(P_{10q}^5) \leq \frac{5}{2}k^2 + 1$.

<u>Case 3</u> $n \equiv 2(\text{mod } 10)$ $(rn(P_n^5) \leq \frac{5}{2}k^5 - \frac{1}{2})$

When $n \equiv 2$ (mod 10) $\Rightarrow \exists\, q \in \mathbb{N}$ s.t. $n = 10q+2 \Rightarrow k = diam(P_n^5) = 2q+1$. So $n = 12, 22, 32, \ldots$. For this case we were able to find a pattern where the span matched the general lower bound found earlier. The easiest way to describe the pattern we found is by looking at the levels of the vertices.

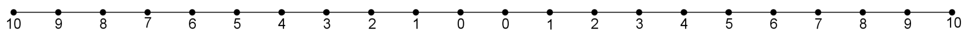

Figure 15: The levels of the vertices of P_{22}

Take P_{22}^5 for example, we start with the left vertex with level 0. Then we move to the right vertex with level 10 followed by the left vertex with level 5. Now we go to the right vertex with level 5 then the left vertex with level 10. Next we go to the right vertex with level 1 followed by the left vertex with level 9. Then we go to the right vertex with level 6 followed by the left vertex with level 4, etc. . . . So using the same method for representing the pattern in case 1,$n \equiv 0(mod\ 10)$ the pattern for labeling P_{22}^5 looks as follows:
L0-R10-L5-R5-L10-R1-L9-R6-L4- R2-L8-R7-L3-R3-L7-R8-L2-R4-L6-R9-L1-R0
After close examination we noticed that the pattern for P_{22}^5 can be re-written as:
L0-R($5q$)-L5-R($5q - 5$)-L10-R1-L($5q - 1$)-R6-($5q - 6$)-R2-L($5q - 2$)-R7
-L($5q - 7$)-R3-L($5q - 3$)-R8-L($5q - 8$)-R4-L($5q - 4$)-R9-L1-R0.
We also found that this pattern can be extended for all P_n^5 when $n = 10q + 2$. So in general, the pattern is:

$$L0 \xrightarrow{5q+1} R(5q) \xrightarrow{5q+6} L5 \xrightarrow{5q+1} R(5q-5) \xrightarrow{5q+6} L10 \xrightarrow{5q+1} R(5q-10) \xrightarrow{5q+6} L15 \xrightarrow{5q+1} R(5q-15) \xrightarrow{5q+6} \ldots \xrightarrow{5q+6} L(5q-5) \xrightarrow{5q+1} R5 \xrightarrow{5q+6} L(5q)$$
$$\xrightarrow{5q+2} R1 \xrightarrow{5q+1} L(5q-1) \xrightarrow{5q+6} R6 \xrightarrow{5q+1} L(5q-6) \xrightarrow{5q+6} R11 \xrightarrow{5q+1} L(5q-11) \xrightarrow{5q+6} \ldots \xrightarrow{5q+6} R(5q-9) \xrightarrow{5q+1} L9 \xrightarrow{5q+6} R(5q-4) \xrightarrow{5q+1} L4$$
$$\xrightarrow{7} R2 \xrightarrow{5q+1} L(5q-2) \xrightarrow{5q+6} R7 \xrightarrow{5q+1} L(5q-7) \xrightarrow{5q+6} R12 \xrightarrow{5q+1} L(5q-12) \xrightarrow{5q+6} \ldots \xrightarrow{5q+6} R(5q-8) \xrightarrow{5q+1} L8 \xrightarrow{5q+6} R(5q-3) \xrightarrow{5q+1} L3$$
$$\xrightarrow{7} R3 \xrightarrow{5q+1} L(5q-3) \xrightarrow{5q+6} R8 \xrightarrow{5q+1} L(5q-8) \xrightarrow{5q+6} R13 \xrightarrow{5q+1} L(5q-13) \xrightarrow{5q+6} \ldots \xrightarrow{5q+6} R(5q-7) \xrightarrow{5q+1} L7 \xrightarrow{5q+6} R(5q-2) \xrightarrow{5q+1} L2$$
$$\xrightarrow{7} R4 \xrightarrow{5q+1} L(5q-4) \xrightarrow{5q+6} R9 \xrightarrow{5q+1} L(5q-9) \xrightarrow{5q+6} R14 \xrightarrow{5q+1} L(5q-14) \xrightarrow{5q+6} \ldots \xrightarrow{5q+6} R(5q-6) \xrightarrow{5q+1} L6 \xrightarrow{5q+6} R(5q-1) \xrightarrow{5q+1} L1 \xrightarrow{2} R0$$

Thus $x_1 = L0$, $x_2 = R(5q)$,$x_3 = L5$,$\ldots$,$x_n = x_{10q+2} = R0$. The values above the arrow show the distances between the two consecutive vertices. By Lemma 5, f is a radio-labeling for P^5_{10q+2}. Observe from above, there are four possible distances in P^5_{10q+2} between consecutive vertices, 1, 2, $q+1$, and $q+2$, with the number of occurrences 1, 3, $5q+1$, and $5q-4$ respectively. It follows by direct calculation (note that $q = \frac{k-1}{2}$) that

$$f(x_{10q+2}) = (10q+1)(k+1) - \sum_{i=1}^{10q+1} d(x_i, x_{i+1}) = \tfrac{5}{2}k^2 - \tfrac{1}{2}$$

As an example, the following figures show our pattern for the first two cases for $n = 10q+2$:

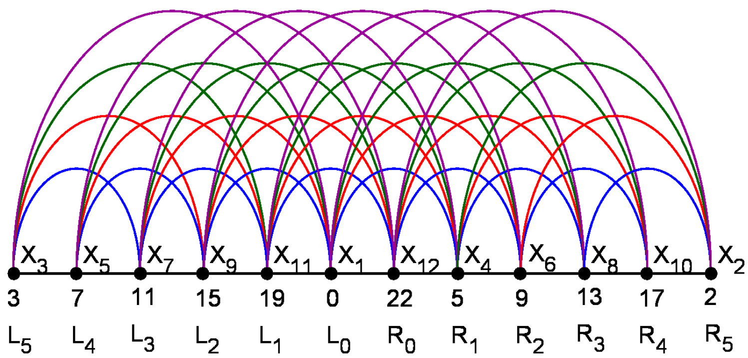

Figure 16: A radio-labeling of P^5_{12}

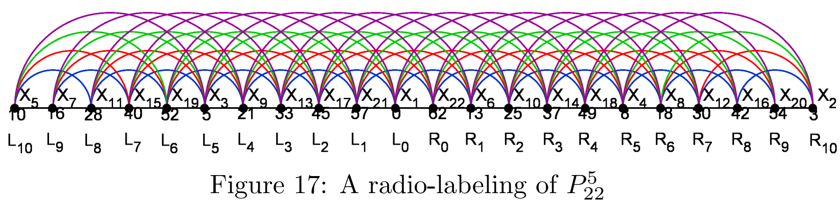

Figure 17: A radio-labeling of P^5_{22}

<u>Case 4</u> $n \equiv 3 (\text{mod } 10)$ $(rn(P^5_n) \leq \frac{5}{2}k^5 + \frac{1}{2})$

When $n \equiv 3$ (mod 10) $\Rightarrow \exists\, q \in \mathbb{N}$ s.t. $n = 10q+2 \Rightarrow k = diam(P^5_n) = 2q+1$. So

$n = 13, 23, 33, \ldots$. For this case we were able to find a pattern where the span matched the general lower bound found earlier. The easiest way to describe the pattern we found is by looking at the levels of the vertices.
Take P_{23}^5 for example, we start with the center vertex. Then we move to the right vertex with level 11. followed by the left vertex with level 5. Now we go to the right vertex with level 6 then the left vertex with level 10. Next we go to the right vertex with level 1 followed by the left vertex with level 7. Then we go to the right vertex with level 4 followed by the left vertex with level 2, etc.... So using the same method for representing the pattern in case 1,$n \equiv 0(mod\ 10)$ the pattern for labeling P_{23}^5 looks as follows:
C-R11-L5-R6-L10-R1-L7-R4-L2-R9-L3-R8-L8-R3-L9-R2-L4-R7-L11-R5-L6-R10-L1
Where C represents the center, which has a level of 0. After close examination we noticed that the pattern for P_{22}^5 can be re-written as:
C-R$(5q+1)$-L5-R$(5q-4)$-L10-R1-L(5q-3) $- R4 - (5q-8)$-R9-L3-R(5q-2)-L8-R3-L(5q-1)-R2-L(5q-6)R(5q-3)-L(5q+1)-R5-L(5q-4)-R10-L1
We also found that this pattern can be extended for all P_n^5 when $n = 10q + 3$. So in general, the pattern is:

$$C \xrightarrow{5q+1} R(5q+1) \xrightarrow{5q+6} L5 \xrightarrow{5q+1} R(5q-4) \xrightarrow{5q+6} L10 \xrightarrow{5q+1} R(5q-9) \xrightarrow{5q+6} L15 \xrightarrow{5q+1} R(5q-14) \xrightarrow{5q+6} \ldots \xrightarrow{5q+6} L(5q-5) \xrightarrow{5q+1} R6 \xrightarrow{5q+6} L(5q) \xrightarrow{5q+1} R1$$
$$\xrightarrow{5q-2} L(5q-3) \xrightarrow{5q+1} R4 \xrightarrow{5q-4} L(5q-8) \xrightarrow{5q+1} R9 \xrightarrow{5q-4} L(5q-13) \xrightarrow{5q+1} \ldots \xrightarrow{5q+1} R(5q-11) \xrightarrow{5q-4} L7 \xrightarrow{5q+1} R(5q-6) \xrightarrow{5q-4} L2 \xrightarrow{5q+1} R(5q-1)$$
$$\xrightarrow{5q+2} L3 \xrightarrow{5q+1} R(5q-2) \xrightarrow{5q+6} L8 \xrightarrow{5q+1} R(5q-7) \xrightarrow{5q+6} L13 \xrightarrow{5q+1} R(5q-12) \xrightarrow{5q+6} \ldots \xrightarrow{5q+6} L(5q-7) \xrightarrow{5q+1} R13 \xrightarrow{5q+6} L(5q-2) \xrightarrow{5q+1} R3$$
$$\xrightarrow{5q+2} L(5q-1) \xrightarrow{5q+1} R2 \xrightarrow{5q-4} L(5q-6) \xrightarrow{5q+1} R7 \xrightarrow{5q-4} L(5q-11) \xrightarrow{5q+1} \ldots \xrightarrow{5q+1} R(5q-13) \xrightarrow{5q-4} L9 \xrightarrow{5q+1} R(5q-8) \xrightarrow{5q-4} L4 \xrightarrow{5q+1} R(5q-3)$$
$$\xrightarrow{10q-2} L(5q+1) \xrightarrow{5q+6} R5 \xrightarrow{5q+1} L(5q-4) \xrightarrow{5q+6} R10 \xrightarrow{5q+1} L(5q-9) \xrightarrow{5q+6} R15 \xrightarrow{5q+1} L(5q-14) \xrightarrow{5q+6} \ldots \xrightarrow{5q+6} R(5q-10) \xrightarrow{5q+1} L6 \xrightarrow{5q+6} R(5q) \xrightarrow{5q+1} L1$$

Thus $x_1 = C$, $x_2 = R(5q+1)$,$x_3 = L5$,...,$x_n = x_{10q+3} = L1$. The values above the arrow show the distances between the two consecutive vertices. By Lemma 5, f is a radio-labeling for P_{10q+3}^5. Observe from above, there are four possible distances in P_{10q+3}^5 between consecutive vertices, q, $q+1$, $q+2$, and $2q$, with the number of occurrences $2q-1$, $5q+3$, $3q-1$, and 1 respectively. It follows by direct calculation (note that $q = \frac{k-1}{2}$) that

$$f(x_{10q+3}) = (10q+2)(k+1) - \sum_{i=1}^{10q+2} d(x_i, x_{i+1}) = \frac{5}{2}k^2 + \frac{1}{2}$$

As an example, the following figures show our pattern for the first two cases for $n = 10q + 3$:

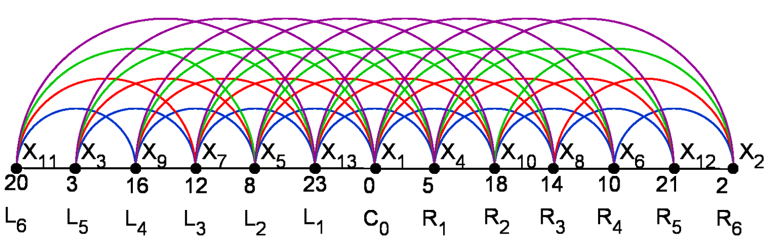

Figure 18: A radio-labeling of P_{13}^5

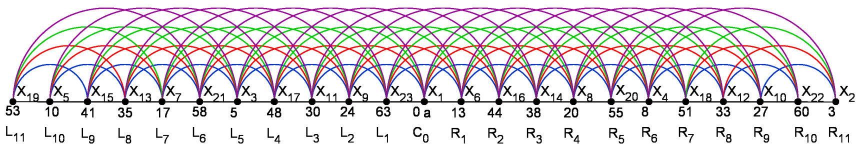

Figure 19: A radio-labeling of P_{23}^5

Case 5 $n \equiv 1 (\text{mod } 10)$

When $n \equiv 1$ (mod 10) $\Rightarrow \exists\ q \in \mathbb{N}$ s.t. $n = 10q + 1 \Rightarrow k = diam(P_n^5) = 2q$. So $n = 11, 21, 31, \ldots$. For this case we were able to find a pattern that matches what we believe to be the sharper lower bound, which we will work on proving in the fall. The easiest way to describe the pattern we found is by looking at the levels of the vertices.
Take P_{21}^5 for example, we start with the left vertex with level 0. Then we move to the right vertex with level 10 followed by the left vertex with level 2. Now we go to the right vertex with level 9 then the left vertex with level 3. Next we go to the right vertex with level 8 followed by the left vertex with level 7. Then we go to the right vertex with level 4 followed by the left vertex with level 6, etc....
So using the same method for representing the pattern in case 1,$n \equiv 0(mod\ 10)$ the pattern for labeling P_{21}^5 looks as follows:
L1-R10-L2-R9-L3-R8-L4-R7-L5-R6-L6-R5-L7-R4-L8-R3-L9-R2-L10-R1-C
Where C represents the center, which has a level of 0. After close examination, we noticed that the pattern for P_{21}^5 can be re-written as:
L1-R($5q$)-L2-R($5q-1$)-L3-R($5q-2$)-L4-R($5q-3$)-L5-($5q-4$)-L6-R($5q-5$)-L7-R($5q-6$)-L8-R(5q-7)-L9-R(5q-8)-L10-R1-C
We also found that this pattern can be extended for all P_n^5 when $n = 10q+1$. So in general, the pattern is:

$$L1 \xrightarrow{5q+1} R(5q) \xrightarrow{5q+2} L2 \xrightarrow{5q+1} R(5q-1) \xrightarrow{5q+2} L3 \xrightarrow{5q+1} R(5q-2) \xrightarrow{5q+2} L4 \xrightarrow{5q+1} R(5q-3) \xrightarrow{5q+2} \ldots \xrightarrow{5q+2} L(5q-1) \xrightarrow{5q+1} R2 \xrightarrow{5q+2} L(5q) \xrightarrow{5q+1} R1 \xrightarrow{1} C$$

Thus $x_1 = L1$, $x_2 = R(5q)$,$x_3 = L2$,...,$x_n = x_{10q+1} = C$. The values above the arrow show the distances between the two consecutive vertices. By Lemma 5, f is a radio-labeling for P_{10q+1}^5. Observe from above, there are two possible distances in P_{10q+1}^5 between consecutive vertices, 1, and $q+1$, with the number of occurrences 1, and $10q-1$ respectively. It follows by direct calculation (note that $q=\frac{k}{2}$) that

$$f(x_{10q+1}) = (10q)(k+1) - \sum_{i=1}^{10q} d(x_i, x_{i+1}) = \frac{5}{2}k^2 + \frac{k}{2} = \frac{5}{2}k^2 + q$$

This the sharpest upper-bound we found, as stated before we will work on a proving this is also the lower-bound in the fall. As an example, the following figures show our pattern for the first two cases for $n = 10q+1$:

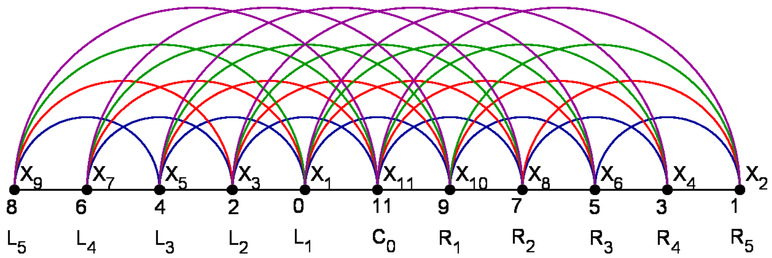

Figure 20: A radio-labeling of P_{11}^5

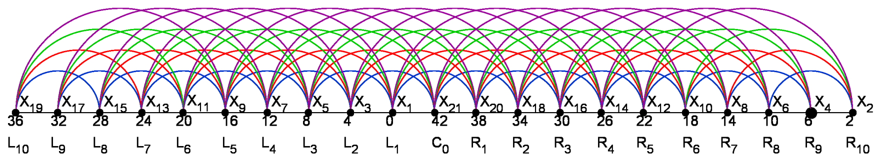

Figure 21: A radio-labeling of P_{21}^5

<u>Case 6</u> $n \equiv 8 (\text{mod } 10)$

When $n \equiv 8$ (mod 10) $\Rightarrow \exists\, q \in \mathbb{N}$ s.t. $n = 10q-2 \Rightarrow k = diam(P_n^5) = 2q$. So $n = 8, 18, 28, 38, \ldots$. Let $G = P_{10q}^5$ and H be the subgraph of G induced by the vertex set $v_1, v_2, v_3, ..., v_{10q-2}$. Then $H \cong P_{10q-2}^5$, $diam(H) = diam(G) = 2q$, and $d_G(u,v) = d_H(u,v)$ for every $u, v \in V(H)$. Let f be a radio-labeling for G, then $f\mid_H$ is also a radio-labeling for H. By Case 1, $rn(P_{10q-2}^5) \le rn(P_{10q}^5) \le \frac{5}{2}k^2 + 1$.

This the sharpest upper-bound we found, as stated before we will work on a proving this is also the lower-bound in the fall.

<u>Case 7</u> $n \equiv 8 (\text{mod } 10)$

When $n \equiv 7$ (mod 10) $\Rightarrow \exists\, q \in \mathbb{N}$ s.t. $n = 10q-3 \Rightarrow k = diam(P_n^5) = 2q$. So $n = 7, 17, 27, 37, \ldots$. Let $G = P_{10q}^5$ and H be the subgraph of G induced by the vertex set $v_1, v_2, v_3, ..., v_{10q-3}$. Then $H \cong P_{10q-3}^5$, $diam(H) = diam(G) = 2q$, and $d_G(u,v) = d_H(u,v)$ for every $u, v \in V(H)$. Let f be a radio-labeling for G, then $f\mid_H$ is also a radio-labeling for H. By Case 1, $rn(P_{10q-3}^5) \le rn(P_{10q}^5) \le \frac{5}{2}k^2 + 1$.

This the sharpest upper-bound we found, as stated before we will work on a proving this is also the lower-bound in the fall.

<u>Cases 8-10</u> We will be working on finding the upper bound for the remaining cases in the fall.

Conclusion

We proved the general lower bound of the radio-number of P_n^5 and proved the upper bound for 7 of the 10 cases. In the fall we will finish our work with the special cases of the lower bound and prove the upper bound for the remaining three cases.

Our results thus far are summed up in the table bellow:

n	k	General Lower Bound	Upper Bound
$10q$	$2q$	$\frac{5}{2}k^2+1$	$\frac{5}{2}k^2+1$
$10q+1$	$2q$	$\frac{5}{2}k^2+1$	$\frac{5}{2}k^2+q$
$10q+2$	$2q+1$	$\frac{5}{2}k^2-\frac{1}{2}$	$\frac{5}{2}k^2-\frac{1}{2}$
$10q+3$	$2q+1$	$\frac{5}{2}k^2+\frac{1}{2}$	$\frac{5}{2}k^2+\frac{1}{2}$
$10q+4$	$2q+1$	$\frac{5}{2}k^2+\frac{1}{2}$	
$10q+5$	$2q+1$	$\frac{5}{2}k^2+\frac{3}{2}$	
$10q+6$	$2q+1$	$\frac{5}{2}k^2+\frac{3}{2}$	
$10q+7$	$2q+2$	$\frac{5}{2}k^2$	$\frac{5}{2}k^2+1$
$10q+8$	$2q+2$	$\frac{5}{2}k^2$	$\frac{5}{2}k^2+1$
$10q+9$	$2q+2$	$\frac{5}{2}k^2+1$	$\frac{5}{2}k^2+1$

The following table gives the results we expect to find by the end of the fall term when we finish the proofs mentioned above.

n	k	General Lower Bound	Upper Bound
$10q$	$2q$	$\frac{5}{2}k^2+1$	$\frac{5}{2}k^2+1$
$10q+1$	$2q$	$\frac{5}{2}k^2+q$	$\frac{5}{2}k^2+q$
$10q+2$	$2q+1$	$\frac{5}{2}k^2-\frac{1}{2}$	$\frac{5}{2}k^2-\frac{1}{2}$
$10q+3$	$2q+1$	$\frac{5}{2}k^2+\frac{1}{2}$	$\frac{5}{2}k^2+\frac{1}{2}$
$10q+4$	$2q+1$	$\frac{5}{2}k^2+\frac{3}{2}$	$\frac{5}{2}k^2+\frac{3}{2}$
$10q+5$	$2q+1$	$\frac{5}{2}k^2+\frac{5}{2}$	$\frac{5}{2}k^2+\frac{5}{2}$
$10q+6$	$2q+1$	$\frac{5}{2}k^2+\frac{5}{2}$	$\frac{5}{2}k^2+\frac{5}{2}$
$10q+7$	$2q+2$	$\frac{5}{2}k^2+1$	$\frac{5}{2}k^2+1$
$10q+8$	$2q+2$	$\frac{5}{2}k^2+1$	$\frac{5}{2}k^2+1$
$10q+9$	$2q+2$	$\frac{5}{2}k^2+1$	$\frac{5}{2}k^2+1$